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A NEW ROTATIONAL METHOD IN FACTOR ANALYSIS

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A new rotational method, particularly appropriate when a positive manifold can be assumed, is presented in theory and in computational detail. It is applied to a five-dimensional factorial matrix in which a simple configuration is known to exist. The new technique is much more efficient in yielding the simple configuration than are the methods formerly used. Landahl's transformations can be combined with this method to advantage.

Since the publication of *The Vectors of Mind,* several new procedures have been tried to shorten the computational work of isolating a simple configuration in a test battery. Some of these procedures have proved to be successful in a number of factorial studies, and it now seems desirable to make them available by publication. Since the new method constitutes a very marked improvement on the older methods, we have entirely abandoned the longer methods that have been previously described. The present method is especially well suited to problems with psychological tests in which a positive manifold can be assumed. This assumption is independent of the existence of a simple configuration. In addition to a description of the new method as regards its theory, we shall give here also a complete numerical example in sufficient detail to serve as a model for the computational work. The new method gives the same solution as the older methods, but it is much more efficient in finding the simple configuration if it exists in the test battery. Eventually even the present method will seem cumbersome with further improvements in technique.

The new method consists, in effect, in taking three-dimensional sections of the configuration instead of two-dimensional sections as before.* But each three-dimensional section is represented in the plane of a drawing, so that solid models are not necessary. With the new rotational method a four-dimensional problem collapses immediately to the simple configuration. We have, therefore, chosen a five-dimensional problem for the numerical example so that several rotations are necessary to reach the solution.

It will be assumed here that the correlational matrix has been factored by the centroid method† or by an equivalent method. Our

* Thurstone, L. L., *Primary Mental Abilities,* p. 73, The University of Chicago Press, 1938.

† Thurstone, L. L., *The Vectors of Mind,* Chapter III, Chicago: The University of Chicago Press, 1935.

problem starts, then, with the rotation of the coordinate axes of the centroid matrix.

In determining the rotations of the reference axes in the older method we plotted on cross section paper every pair of columns of the centroid factorial matrix. A radial streak or band of points on such a diagram indicated the rotation of the axes so as to maximize the number of nearly vanishing projections of the test vectors. It was necessary that the band of points be radial. The new method allows one to draw a line through any straight band of points on the diagram, radial or non-radial, and to determine the new reference vector so that all points on that line have zero projections on the new reference vector. A logical exposition would give the theory first and the numerical example later, but it will probably be clearer to present them together.

A fictitious five-dimensional factorial matrix was constructed as shown in *Table 1*. It represents a positive simple structure as may be expected in a battery of psychological tests. There are 25 fictitious tests. The first test calls for only two primary factors, namely 2 and 5, in that only these two entries in the first row are non-vanishing. Each row can be interpreted in the same manner. The number of zero entries in this table is such that the battery shows a simple structure. The axes implied in *Table 1* were rotated by an arbitrary orthogonal transformation Λ_s as shown in *Table 2*. This transformation was so adjusted that the first axis contained the centroid of the configuration. After rotation the configuration is described in terms of a new arbitrary orthogonal reference frame. The new factorial matrix is shown in *Table 3*. We shall assume that this factorial matrix is given, and that it is our problem to discover the simple configuration by a method that will produce the matrix we started with, namely, *Table 1*. It is as if we were asked to rotate the axes of *Table 3* without knowing anything about the original in *Table 1*. *Table 3* represents a centroid matrix obtained from the intertest correlations.

The new method consists of several distinct operations that can be denoted as follows:

1) The extension of each *test vector* so that its projection on the centroid axis is unity. The termini of all the test vectors then lie in a hyperplane orthogonal to the centroid axis and at unit distance from the origin. The matrix of direction numbers of the extended test vectors will be called the *extended factorial matrix.*

2) An arbitrary orthogonal rotation of the *reference vectors* so that they all make the same angle with the centroid axis. This arbitrary rotation of the reference axes was suggested by Mr. Herbert Landahl, and he has prepared a set of orthogonal transformations to

bring all the axes to the same angle with the centroid axis. Such rotations are not unique, but he has prepared a computationally convenient orthogonal transformation for each order. These will be referred to as Landahl's transformation.* The method can be used without Landahl's transformations, but it is more effective if these transformations are used.

3) The pairs of columns of the extended factorial matrix are plotted as formerly.

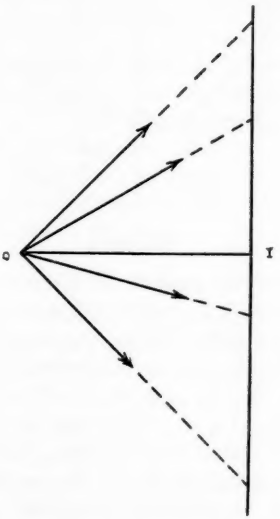

FIGURE 1

4) New reference axes are chosen by these graphs. The process in steps (3) and (4) is repeated until no further rotations are evident.

5) The vectors are reduced to their original length and a factorial matrix prepared. The columns of this factorial matrix represent the primary abilities.

In *Figure 1* several test vectors are represented. The axis *I* is the first centroid axis. The extensions of the test vectors are represented by dotted lines. They are extended so that they all have unit projection on the first centroid axis. The vertical line represents the hyperplane that contains the termini of all the test vectors. This hyperplane is orthogonal to the centroid axis *I*, and it is at unit distance from the origin.

* Landahl, H. D., "Centroid Orthogonal Transformations," *Psychometrika*, current issue.

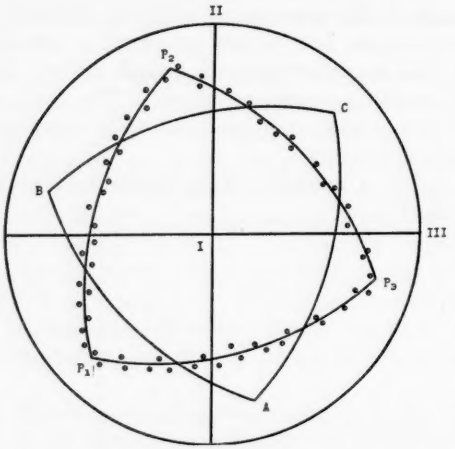

FIGURE 2

The extended factorial matrix E_c is shown in *Table 4*. It is obtained from F_c in *Table 3* by multiplying each row by $1/x$, where x is the first entry in the row. These reciprocals are shown in column D_e of *Table 3*. Then the first entry becomes unity in each row of E_c in *Table 4*. The same multiplier applies to all the entries in a row of F_c.

In *Figure 2* let P_1, P_2, P_3, represent primary trait vectors, and let the small circles represent test vectors, lying inside the unit sphere, that define a simple configuration in the spherical triangle. The example refers to three dimensions, but these principles are general. Let *I* be the first centroid axis which is necessarily inside the configuration of $P_1P_2P_3$. Let *II* and *III* be the other two reference axes in the orthogonal reference frame for the centroid factorial matrix. If we are to look down into the configuration, as it were, along the first centroid axis, it will facilitate the appearance of bounding coordinate hyperplanes if the reference frame *I, II, III,* can be moved so as to enclose the configuration as far as possible. Landahl's transformation rotates the reference frame *I, II, III,* to an arbitrary position *ABC* which is so chosen that the centroid axis *I* of the configuration makes equal angles with *A, B,* and *C*. The problem is then to rotate the reference frame from *I, II, III,* or preferably from *A, B, C,* to the unique position $P_1P_2P_3$ which defines the simple structure in the test battery.

The matrix E_c of *Table 4* is postmultiplied by the Landahl matrix M_0 in *Table 5* in order to obtain the extended matrix E_0 in *Table 5*. This table shows a convenient arrangement of the computations.

There are several minor modifications which a computer will introduce, including check columns and rows which are not shown in this numerical example. The work has been shown here so as to be as clear as possible as regards the principal operations.

The computations thus far are summarized as follows:

1. The given centroid matrix F_c in *Table 3* is written in the extended form E_c of *Table 4*. This operation can be represented formally in the matrix equation,

$$D_e F_c = E_c , \tag{1}$$

where D_e is a diagonal matrix of order $n \times n$ whose entries are the reciprocals of the first entries in the corresponding rows of F_e. In practice one does not write this diagonal matrix. One merely performs the multiplication for each row of F_c.

2. The matrix E_c of *Table 4* is multiplied by Landahl's matrix M_o to produce the extended factorial matrix E_o of *Table 5*. This can be represented formally in the equation

$$E_c M_o = E_o . \tag{2}$$

The matrix E_o has five columns in addition to the first column with entries of unity. These five columns give ten possible pairs. Each pair of columns is plotted on a separate graph. Four of these are shown in *Figure 3*. The purpose of the graphs is to find, by several adjustments, the bounding coordinate hyperplanes. On the graphs we look for linear arrangements of points.

In the graph for columns C and D a line has been drawn through or near fourteen points. It has been denoted the D plane because the line is more nearly orthogonal to the axis D than to C. The naming of the planes is of no consequence except when the adjustments are small in the later trials. Then it is well to retain the identity of each plane in its notation.

We write the equation of the line in the CD graph. This is done by inspection of the intercepts and the slope or in any other convenient way. For this graph the equation can be written by inspection in the form

$$.68C + 1.00D - .26 = 0 . \tag{3}$$

These coefficients are written in the column D of the matrix S_1 in *Table 5*. Consider a vector which is a linear combination of the vectors C and D and the centroid vector I, namely,

$$\Lambda_D = .68C + 1.00D - .26I . \tag{4}$$

The coefficients of (4) are identical with those of (3) except that the

numerical term is taken as the coefficient of the vector I. The projections of all the test vectors on the centroid vector are now unity. Equation (4) represents a new reference vector Λ_D. All of the fourteen points on or near the line on the graph CD will have vanishing or nearly vanishing projections on the new reference vector Λ_D. Furthermore, if the line on the graph is bounding so that all the points lie in or on one side of the line, then the projections on the new vector Λ_D will be like-signed. Points on opposite sides of a line in the graph will have projections of unlike signs. Since we are looking for a positive orthogonal manifold, we choose trial lines on the graphs that bound the configuration of points.

If the signs of the coefficients in (4) are reversed, the effect will be to define Λ_D in the opposite direction. The absolute values of all the projections will remain the same, but they will have reversed signs. It is convenient to know that if the coefficients in (4) are written with the numerical term in (3) positive, then the points in the graph on the same side of the line with the origin will have positive projections, and points on the other side of the line will have negative projections. Inspection of the graph CD shows that the points that are not in or near the line are on the opposite side from the origin. Hence the coefficient of the vector I is written with negative sign. This has been done in column D of S_1 in *Table 5*. If the origin had been on the same side with the points that are not on the line in CD, the coefficients of (4) would have been reversed in sign so as to be —.68, —1.00, and +.26. There is no harm if this matter of signs is ignored because the whole column D in the next factorial matrix is then negative. When a column of negative signs is discovered, one has merely to reverse the signs of the direction cosines of the corresponding reference vector to make all of the projections positive.

In the graph AD we have drawn a line close to twelve or more points. It is about equally separated from the first reference vectors A and D.

It is arbitrarily denoted the A plane. There is going to be only one noticeably negative projection on this reference vector Λ_A.

The equation of the line may be written

$$1.00A + 1.03D - 1.50 = 0. \tag{5}$$

In this graph the origin lies on the same side of the line as the majority of the points. Hence the equation is written with positive constant term, namely,

$$-1.00A - 1.03D + 1.50 = 0, \tag{6}$$

and then the new reference vector Λ_A will be

$$\Lambda_A = -1.00A - 1.03D + 1.50I. \tag{7}$$

These coefficients are entered in column A of S_1 in *Table 5.*

In the graph BD a line has been drawn so as to bound the configuration and so as to lie near a large number of points. Its equation is written in terms of the variables B and D with a numerical term. The latter is taken as the coefficient of the vector I, and the three coefficients are written in column B of S_1 in *Table 5.*

The graph CE shows two possibilities, and hence two lines are drawn on this graph so as to bound the configuration as far as possible and so as to lie near a large number of points. These are denoted the C plane and the E plane. Their equations are written, and the numerical term is in each case taken as the coefficient of the vector I. The coefficients are written in the columns C and E in S_1 of *Table 5.* With a little practice this operation of filling in the entries in the matrix S_1 becomes easily routinized.

The columns of S_1 show the direction numbers of vectors that are collinear with the new unit reference vectors. The length of each of the five column vectors in S_1 must be determined. But the column vectors in S_1 are usually not orthogonal. Their lengths cannot be determined by the sum of the squares of the entries in each column of S_1 because the reference frame is not orthogonal. These vectors are, therefore, referred to the orthogonal centroid frame of E_c. From (2) we have E_o by the transformation M_o. Hence the column vectors in S_1 can be written in terms of the orthogonal frame of E_c by the transformation

$$M_o S_1 = L_1 , \tag{8}$$

where columns of L_1 show the same vectors as the columns of S_1. The column vectors of L_1 are referred to the orthogonal frame of E_c. Hence the lengths of the column vectors of S_1 can be found from the corresponding columns of L_1. The sum of the squares of the elements in each column of L_1 are recorded and also the reciprocal of the square root of that sum. The latter is the multiplier by which the corresponding vector in S_1 can be reduced to a unit vector. Let the multipliers be denoted d_r and let these be the elements of a diagonal matrix D_1. Then

$$L_1 D_1 = M_1 , \tag{9}$$

in which the columns of M_1 in *Table 6* show the direction cosines of the new unit reference vectors Λ_r in terms of the orthogonal reference frame.

Applying the same stretching factors in D_1 to the column vectors in S_1, we have

$$S_1 D_1 = T_1 , \tag{10}$$

which is the transformation that carries the extended factorial matrix E_0 into the next factorial matrix E_1. Then

$$E_0T_1 = E_1, \tag{11}$$

which is the starting point for the next cycle of rotations. The factorial matrix E_1 can also be obtained directly from the extended centroid matrix E_c, so that

$$E_cM_1 = E_1. \tag{12}$$

The computational work can be summarized as follows:

1. When the entries in S_1 have been obtained from the graph, perform the matrix multiplication (8).

2. Sum the squares of the entries in each column of L_1 and determine the reciprocal of the square root of each of these sums. The modulus d_r is recorded under each column.

3. Multiply each column of S_1 by the corresponding modulus d_r to obtain T_1, as shown in *Table 5*. This step is shown formally in (10). Also determine M_1 from (9).

4. Multiply the matrix E_0 by T_1 to obtain E_1 in *Table 6*. This step is shown in (11).

Table 6 shows the computations for the second cycle of rotations. The matrices E_1 and M_1 have already been obtained, as shown in (9) and in (11). First determine the scalar products in M'_1M_1. The side entries in this matrix show the cosines of the angular separations of the new reference vectors. If two of the new vectors are nearly collinear, a new plane should be found in the first set of graphs. One may carry more planes than there are dimensions, but in such a case they will converge so that before the final rotations are made there will be usually fewer meaningful planes than there are dimensions.

The pairs of columns of E_1 are plotted. Five of the diagrams are shown in *Figure 4*. Now the bounding planes are more clearly defined, and the rotations are all smaller in angular magnitude. The identity of each plane is easily retained. From the graphs we determine the entries in the matrix S_2. The matrix multiplication,

$$M_1S_2 = L_2, \tag{13}$$

corresponds to (8) for the first cycle of rotations. The reciprocals of the lengths of the column vectors in L_2 are recorded as d_r. These multipliers on L_2 give

$$L_2D_2 = M_2, \tag{14}$$

which is recorded on the next data sheet. The multiplication,

$$S_2D_2 = T_2, \tag{15}$$

corresponds to (10) for the first cycle. Finally,

$$E_1 T_2 = E_2 , \tag{16}$$

which is recorded on the data sheet for the next cycle.

For the third cycle we condense our description to the matrix S_3 in *Table 7* and the graphs in *Figure 5*. These graphs show still smaller remaining movements. The fourth cycle has the matrix S_4 in *Table 8*, which is determined from the graphs in *Figure 6*. These graphs show the planes to be so nearly in place that it is not worth the trouble to carry further approximations. They would affect only the second decimal place in the factorial matrix.

The last step is to reduce each of the extended vectors to its original length. This is done by dividing each row of E_4 by the stretching factor D_e which produced E_e from F_e. The final matrix F_4 can be obtained from the given matrix F_e by the transformation

$$F_e \Lambda_4 = F_4 , \tag{17}$$

where Λ_4 is an $r \times r$ matrix. If the first column of M_4 is deleted, the remaining matrix is Λ_4. It is shown in *Table 9*. The columns of Λ_4 show the direction cosines of the unit reference vectors which define the simple configuration. They are expressed in terms of the orthogonal reference frame of the initial centroid factorial matrix F_e. The scalar products of columns of Λ_4 are shown in the matrix $\Lambda'_4 \Lambda_4$. The side entries in this symmetric matrix show the correlations between the primary factors in the fictitious test battery. These are practically zero. The final matrix F_4 of *Table 9* is shown in the graphs of *Figures 7* and *8*. *Table 9* is compared with *Table 1*, and it is seen that the approximation to the original simple structure is very close. Further rotations would make the two matrices identical within any required degree of accuracy, and the correlations between the primaries would, of course, also become exactly zero. The present example has been carried farther than would be necessary in any experimental problem.

The plots are made with orthogonal axes on the graphs even though the reference vectors are oblique. Angles between reference vectors that are less than 45 degrees are plotted to scale on the graphs. But it is not often necessary to represent on the graphs the true angle between the reference vectors. On each graph in this numerical example the cosine of the angle between the two reference vectors is recorded, but all the graphs are drawn with orthogonal axes as shown in the figures.

In this method each of the coordinate planes is given the opportunity to find its own location independently of the other planes. In

general, this gives an oblique set of reference vectors unless the underlying order is orthogonal. When the configuration turns out to be orthogonal without imposing orthogonality, it can be taken as good evidence of an underlying physical order. But we must consider the possibility of oblique configurations that are themselves of considerable interest. Under certain conditions we should expect differential maturation of the factors to be responsible for an oblique configuration. At any rate, the method here described does not impose orthogonality on the system. But orthogonality can be maintained in the successive rotations by watching the cosines in the product matrices $M'M$ if orthogonality is to be imposed.

TABLE 1. A fictitious simple configuration

	P_1	P_2	P_3	P_4	P_5	h^2
1	.0	.3	.0	.0	.8	.73
2	.0	.9	.0	.0	.0	.81
3	.5	.5	.0	.5	.0	.75
4	.5	.7	.0	.0	.0	.74
5	.0	.3	.3	.0	.7	.67
6	.0	.4	.6	.4	.0	.68
7	.0	.0	.8	.0	.0	.64
8	.0	.5	.7	.0	.0	.74
9	.0	.0	.6	.6	.0	.72
10	.0	.0	.4	.6	.5	.77
11	.6	.0	.5	.4	.0	.77
12	.7	.3	.3	.0	.0	.67
13	.0	.0	.0	.0	.9	.81
14	.0	.8	.0	.4	.0	.80
15	.6	.0	.5	.0	.3	.70
16	.6	.0	.6	.0	.0	.72
17	.0	.0	.7	.0	.5	.74
18	.4	.0	.0	.0	.7	.65
19	.8	.0	.0	.0	.0	.64
20	.0	.0	.0	.6	.6	.72
21	.0	.0	.0	.8	.0	.64
22	.3	.0	.0	.4	.7	.74
23	.0	.5	.0	.5	.5	.75
24	.7	.3	.0	.0	.3	.67
25	.3	.0	.0	.8	.0	.73

TABLE 2. An arbitrary orthogonal transformation A_a

	I	*II*	*III*	*IV*	*V*
P_1	.447	—.148	—.092	—.832	—.282
P_2	.409	.543	.595	—.084	.420
P_3	.447	—.431	—.375	.128	.678
P_4	.447	.554	—.554	.307	—.307
P_5	.484	—.438	.438	.438	—.438

TABLE 3. A centroid matrix F_c of the fictitious test battery

	I	*II*	*III*	*IV*	*V*	D_e
1	.510	—.188	.529	.325	—.224	1.9608
2	.368	.489	.536	—.076	.378	2.7174
3	.652	.474	—.026	—.304	—.084	1.5337
4	.510	.306	.370	—.475	.153	1.9608
5	.596	—.273	.373	.320	.023	1.6779
6	.611	.180	—.209	.166	.452	1.6367
7	.358	—.345	—.300	.102	.542	2.7933
8	.517	—.030	.035	.048	.685	1.9342
9	.536	.074	—.557	.261	.223	1.8657
10	.689	—.059	—.263	.454	—.132	1.4514
11	.670	—.083	—.464	—.312	.047	1.4925
12	.570	—.070	.002	—.569	.132	1.7544
13	.436	—.394	.394	.394	—.394	2.2936
14	.506	.656	.254	.056	.213	1.9763
15	.637	—.436	—.111	—.304	.038	1.5699
16	.536	—.347	—.280	—.422	.238	1.8657
17	.555	—.521	—.044	.309	.256	1.8018
18	.518	—.366	.270	—.026	—.419	1.9305
19	.358	—.118	—.074	—.666	—.226	2.7933
20	.559	.070	—.070	.447	—.447	1.7889
21	.358	.443	—.443	.246	—.246	2.7933
22	.652	—.129	.057	.180	—.514	1.5337
23	.670	.330	.240	.330	—.162	1.4925
24	.581	—.072	.246	—.476	—.203	1.7212
25	.492	.399	—.471	—.004	—.330	2.0325

TABLE 4. Extended factorial matrix E_c

	I	*II*	*III*	*IV*	*V*
1	1.000	—.369	1.037	.637	—.438
2	1.000	1.329	1.457	—.207	1.027
3	1.000	.727	—.040	—.466	—.129
4	1.000	.600	.725	—.931	.300
5	1.000	—.458	.626	.537	.039
6	1.000	.295	—.342	.272	.740
7	1.000	—.964	—.838	.285	1.514
8	1.000	—.058	.068	.093	1.325
9	1.000	.138	—1.039	.487	.416
10	1.000	—.086	—.382	.659	—.192
11	1.000	—.124	—.693	—.466	.070
12	1.000	—.123	.004	—.998	.232
13	1.000	—.904	.904	.904	—.904
14	1.000	1.296	.502	.111	.421
15	1.000	—.684	—.174	—.477	.060
16	1.000	—.647	—.522	—.787	.444
17	1.000	—.939	—.079	.557	.461
18	1.000	—.707	.521	—.050	—.809
19	1.000	—.330	—.207	—1.860	—.631
20	1.000	.125	—.125	.800	—.800
21	1.000	1.237	—1.237	.687	—.687
22	1.000	—.198	.087	.276	—.788
23	1.000	.493	.358	.493	—.242
24	1.000	—.124	.423	—.819	—.349
25	1.000	.811	—.957	—.008	—.671

TABLE 5. Computations for first cycle of rotations

Extended factorial matrix E_0

	I	A	B	C	D	E
1	1.000	.118	1.427	.749	—.341	.280
2	1.000	1.636	1.410	—.440	.538	—.913
3	1.000	1.097	.250	—.084	.394	.577
4	1.000	.984	.941	—.657	.694	.271
5	1.000	.037	1.092	.807	.176	.122
6	1.000	.710	.085	.702	.892	—.154
7	1.000	—.413	—.061	1.137	1.860	—.282
8	1.000	.395	.518	.516	1.338	—.534
9	1.000	.571	—.483	1.114	.812	.224
10	1.000	.370	.135	1.115	.173	.443
11	1.000	.336	—.125	.296	.915	.815
12	1.000	.337	.477	—.340	1.044	.718
13	1.000	—.360	1.431	1.126	—.619	.658
14	1.000	1.607	.591	.103	.265	—.332
15	1.000	—.165	.449	.261	.887	.804
16	1.000	—.132	.140	.101	1.379	.750
17	1.000	—.393	.589	1.135	.778	.126
18	1.000	—.185	1.056	.413	—.097	1.046
19	1.000	.154	.341	—.939	.894	1.785
20	1.000	.558	.311	1.107	—.436	.694
21	1.000	1.553	—.902	1.089	—.237	.732
22	1.000	.270	.567	.692	—.204	.911
23	1.000	.888	.646	.634	—.139	.203
24	1.000	.336	.842	—.317	.439	.933
25	1.000	1.173	—.563	.535	.071	1.020

M_0

	I	A	B	C	D	E
I	1.000	.447	.447	.447	.447	.447
II	.000	.894	—.224	—.224	—.224	—.224
III	.000	.000	.866	—.289	—.289	—.289
IV	.000	.000	.000	.816	—.408	—.408
V	.000	.000	.000	.000	.707	—.707

M'_0M_0

	I	A	B	C	D	E
I	1.000					
A	.447	.999				
B	.447	.000	1.000			
C	.447	.000	.000	.999		
D	.447	.000	.000	.001	1.000	
E	.447	.000	.000	.001	.000	1.000

S_1

	I	A′	B′	C′	D′	E′
I	1.00	1.50	1.21	1.71	—.26	.59
A		—1.00				
B			—1.00			
C				—1.00	.68	—.73
D		—1.03	—.68		1.00	
E				—1.48		1.00

L_1

	I	A′	B′	C′	D′	E′
I	1.000	.593	.459	.601	.491	.711
II	0	—.663	.376	.556	—.376	—.060
III	0	.298	—.669	.717	—.485	—.078
IV	0	.420	.277	—.212	.147	—1.004
V	0	—.729	—.481	1.046	.707	—.707
d_r	1.000	.793585	.950139	.656039	.936939	.703063

T_1

	I	A	B	C	D	E
I	1.000	1.190	1.150	1.122	—.244	.415
A		—.794				
B			—.950			
C				—.656	.637	—.513
D		—.817	—.646		.937	
E				—.971		.703

TABLE 6. Computations for second cycle of rotations

Extended factorial matrix E_1

	I	A	B	C	D	E
1	1.000	1.375	.015	.359	—.086	.228
2	1.000	—.549	—.537	2.297	—.020	—.001
3	1.000	—.003	.658	.617	.072	.864
4	1.000	—.158	—.192	1.290	—.012	.943
5	1.000	1.017	—.001	.474	.435	.087
6	1.000	—.103	.493	.811	1.039	—.053
7	1.000	—.002	.006	.650	2.223	—.367
8	1.000	—.217	—.206	1.302	1.338	—.225
9	1.000	.073	1.084	.174	1.226	.001
10	1.000	.755	.910	—.040	.628	.154
11	1.000	.176	.678	.136	.802	.836
12	1.000	.069	.022	.648	.518	1.094
13	1.000	1.982	.190	—.256	—.107	.300
14	1.000	—.302	.417	1.377	.070	.129
15	1.000	.596	.150	.170	.753	.846
16	1.000	.168	.126	.327	1.112	.890
17	1.000	.866	.088	.255	1.208	—.079
18	1.000	1.416	.209	—.164	—.072	.938
19	1.000	.337	.249	.005	—.004	2.152
20	1.000	1.103	1.136	—.278	.053	.335
21	1.000	.151	2.160	—.303	.228	.371
22	1.000	1.142	.743	—.217	.006	.700
23	1.000	.598	.626	.509	.030	.232
24	1.000	.565	.067	.424	—.035	1.234
25	1.000	.201	1.639	—.219	.163	.858

M_1

	I	A	B	C	D	E
I	1.000	.471	.436	.394	.460	.500
II	0	—.526	.357	.365	—.352	—.042
III	0	.236	—.636	.470	—.454	—.055
IV	0	.333	.263	—.139	.138	—.706
V	0	—.579	—.457	.686	.662	—.497

M'_1M_1

	I	A	B	C	D	E
I	1.000					
A	.471	1.000				
B	.436	.220	1.000			
C	.394	—.339	—.347	.999		
D	.460	—.043	.097	.274	.999	
E	.500	.297	.279	—.087	—.157	1.000

S_2

	I	A′	B′	C′	D′	E′
I	1.00	—.21	—.18	.24	.04	—.25
A		1.00				
B			1.00		—.13	
C		.33	.31	1.00		
D				—.41	1.00	.28
E						1.00

L_2

	I	A′	B′	C′	D′	E′
I	1.000	.391	.378	.445	.443	.379
II	0	—.406	.470	.509	—.398	—.141
III	0	.391	—.490	.656	—.371	—.182
IV	0	.287	.220	—.196	.104	—.667
V	0	—.353	—.244	.415	.721	—.312
d_r	1.000	1.214845	1.185263	.954294	.988718	1.163358

T_2

	I	A	B	C	D	E
I	1.000	—.255	—.213	.229	.040	—.291
A		1.215				
B			1.185		—.129	
C		.401	.367	.954		
D				—.391	.989	.326
E						1.163

TABLE 7. The matrix S_3

	I	*A'*	*B'*	*C'*	*D'*	*E'*
I	1.00	—.07	—.05	.07	.04	—.10
A		1.00	.06			
B		.12	1.00			
C				1.00	—.05	.16
D					1.00	
E				—.14		1.00

TABLE 8. The matrix S_4

	I	*A'*	*B'*	*C'*	*D'*	*E'*
I	1.00	.05	.03	—.05	—.02	.08
A		1.00			.03	
B			1.00	.07		—.08
C				1.00		
D					1.00	
E		—.07	—.03			1.00

TABLE 9. The simple configuration F_4 after four rotations

	A	*B*	*C*	*D*	*E*
1	.797	.004	.314	—.004	.012
2	—.009	—.011	.900	—.004	.027
3	.007	.496	.488	.005	.515
4	—.007	—.007	.687	.001	.522
5	.696	.006	.312	.297	.008
6	.003	.400	.399	.601	.006
7	—.003	.006	—.002	.801	—.007
8	—.008	—.001	.498	.699	.007
9	.011	.605	.000	.604	—.006
10	.513	.608	.009	.401	—.002
11	.008	.405	—.016	.506	.595
12	—.004	.000	.282	.304	.706
13	.900	.009	.017	—.003	.003
14	.001	.391	.800	—.001	.024
15	.298	.008	—.010	.503	.597
16	—.003	.006	—.016	.604	.593
17	.497	.011	.007	.700	—.006
18	.701	.008	.003	.000	.402
19	.000	.002	—.020	.004	.801
20	.614	.607	.012	.001	.002
21	.020	.801	.001	.004	.000
22	.710	.408	.006	.001	.302
23	.507	.499	.510	—.002	.017
24	.298	.001	.288	.002	.710
25	.019	.802	—.007	.006	.300

TABLE 9 (continued)

Λ_4

	A	*B*	*C*	*D*	*E*
I	.488	.451	.407	.448	.456
II	—.428	.539	.540	—.429	—.128
III	.420	—.559	.605	—.382	—.068
IV	.445	.311	—.056	.123	—.833
V	—.452	—.311	.417	.674	—.277

$\Lambda'_4\Lambda_4$

	A	*B*	*C*	*D*	*E*
A	1.000				
B	.034	1.000			
C	.008	—.011	1.000		
D	—.008	.013	—.006	1.000	
E	.003	.002	.006	—.004	1.000

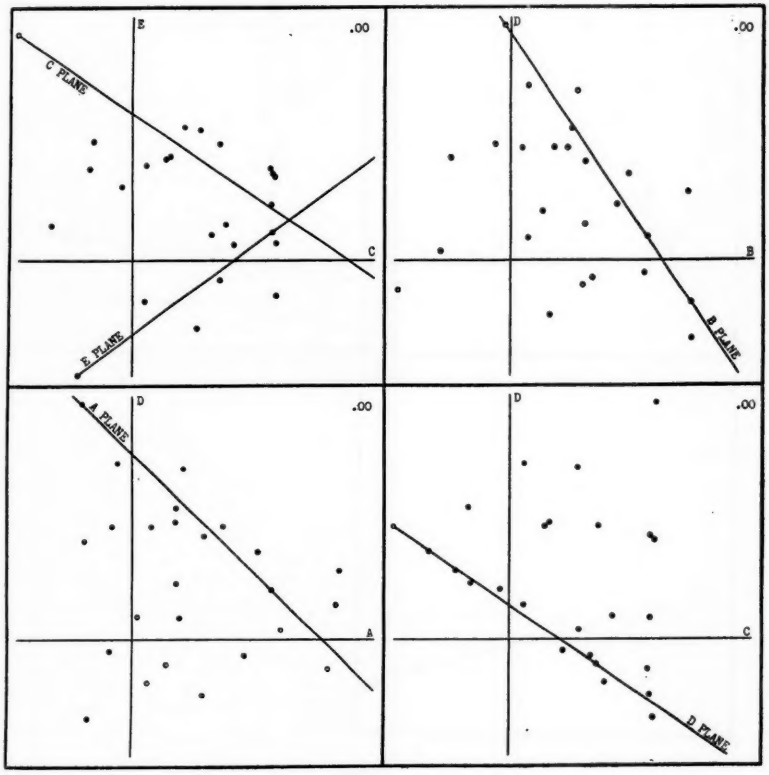

FIGURE 3

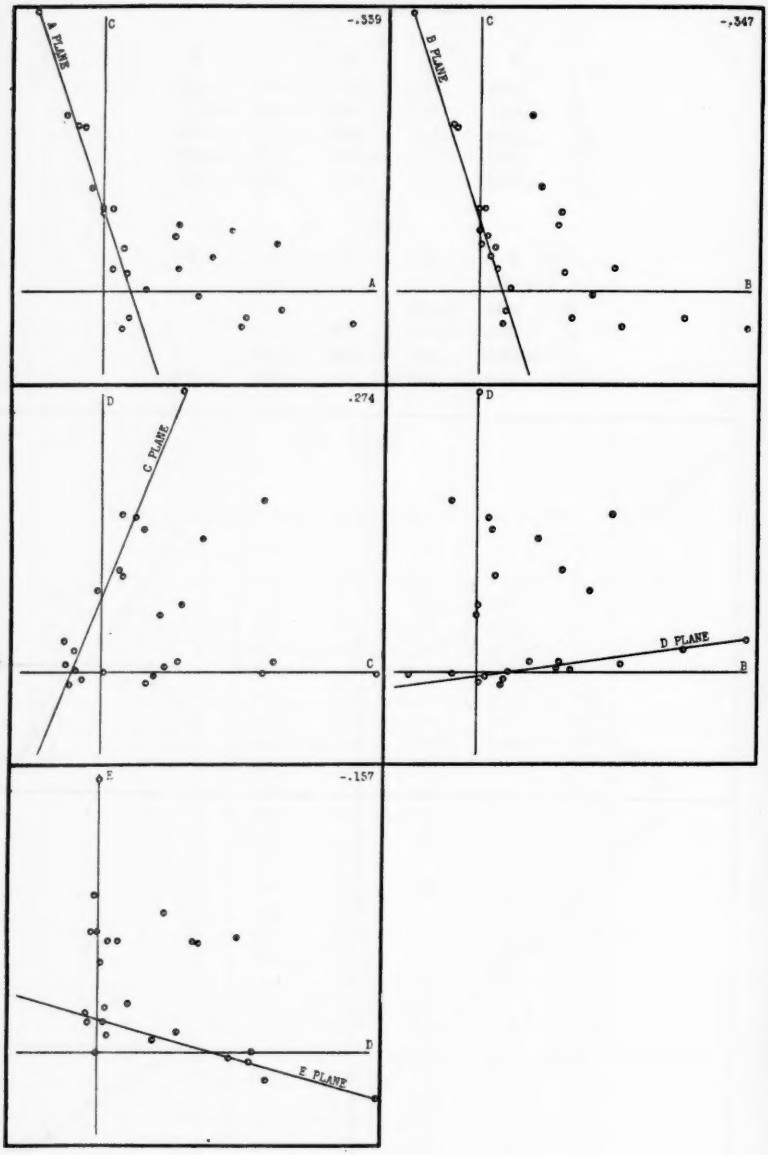
A PLANE
C
-.339
A
B PLANE
C
-.347
B
C PLANE
D
.274
C
D
D PLANE
B
E
-.157
D
E PLANE

FIGURE 4

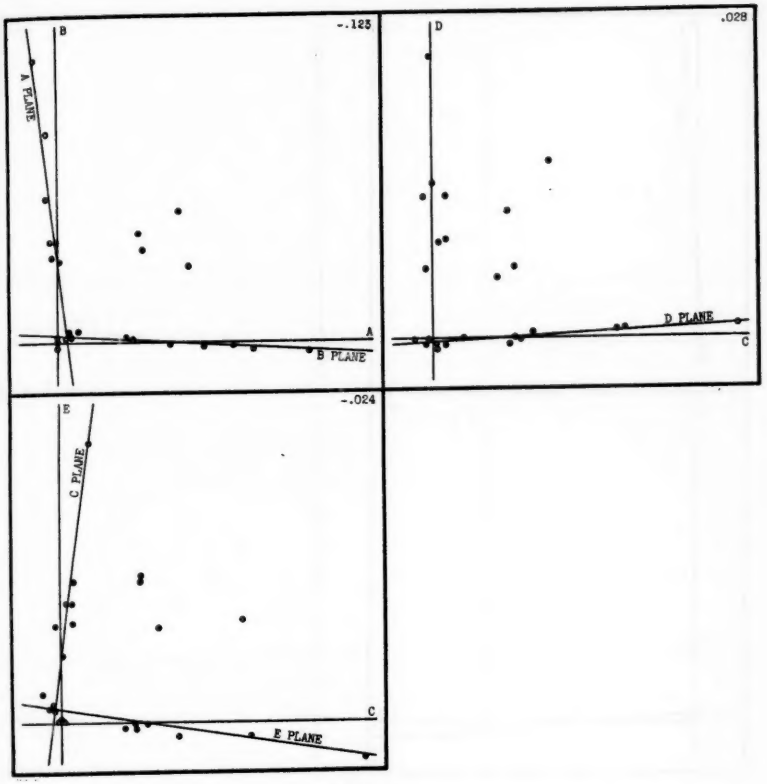

FIGURE 5

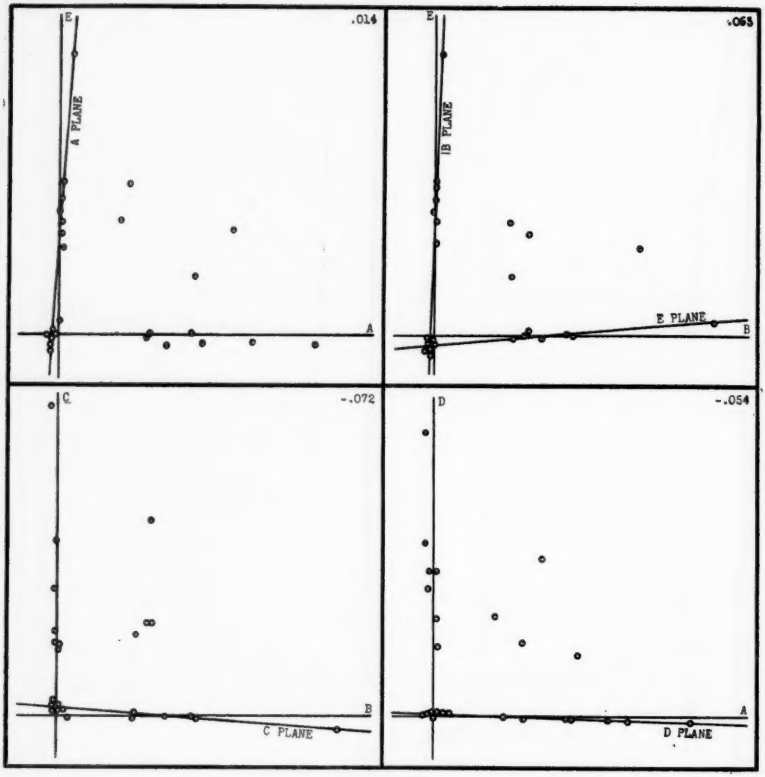
E
.014
A PLANE
A
E
.065
IB PLANE
E PLANE
B
C
-.072
B
C PLANE
D
-.054
A
D PLANE

FIGURE 6

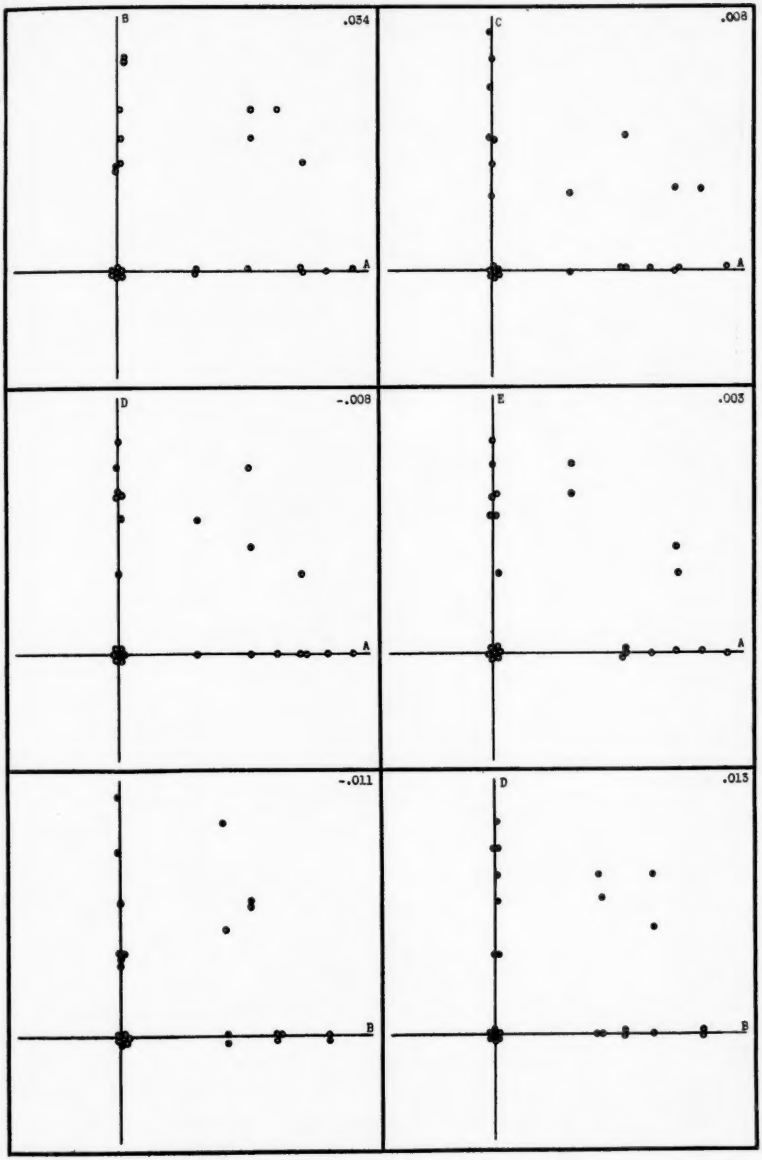

FIGURE 7

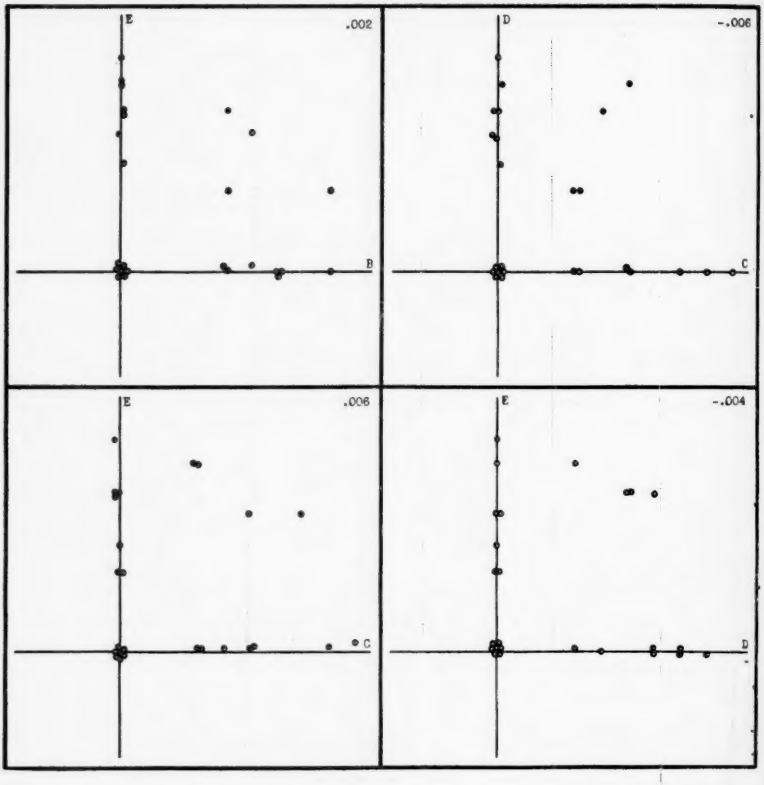

FIGURE 8

PSYCHOMETRIKA—VOL. 3, NO. 4
DECEMBER, 1938

CENTROID ORTHOGONAL TRANSFORMATIONS

H. D. LANDAHL
The University of Chicago

An initial transformation for facilitating the analysis of a factor problem is discussed. Such a transformation is used in the rotational procedure developed by L. L. Thurstone in an accompanying article.

The process of rotating from the centroids of a factored battery of tests to its solution configuration is often facilitated by the introduction of an arbitrary orthogonal rotation applied directly to the factorial matrix, provided the rotation has the property that the sum of its reference vectors is a vector which is collinear with the first centroid of the factorial matrix. There are an infinite number of matrices which satisfy this condition. It is the purpose here to justify the use of such a transformation and to give an appropriate transformation for n dimensions.

In test batteries where the intercorrelations are predominantly positive, the few negatives having small absolute values, most, if not all of the important factors have positive projections on the first centroid. Since these projections are, generally, unknown, and since there is no reason for introducing an oblique reference frame, it is not illogical to take an orthogonal rotation with equal projections on the first centroid. It can be shown that the sum of the reference vectors of such a rotation is a vector collinear with the first centroid.

If the centroid axes are taken in pairs and the projections plotted, the graphs show an almost random scattering of points. Considering that these axes are centroids, such a random scattering is not unexpected; in fact, the factorial matrix is the most unfavorable representation of the data.

Let us present a procedure for determining each element of an $n \times n$ matrix and show that this matrix* is orthogonal, and that the sum of the columns as vectors is a vector collinear with the first centroid.

1. Each element of the first row has the value $1/\sqrt{n}$, where n is the order of the $n \times n$ rotation matrix, that is, the number of dimensions.

* The matrix thus set up is neither unique nor determined by an unique procedure, but is given in this form for convenience only.

2. The remaining elements of the last column, beginning at the last element and going up, are the first $(n - 1)$ terms of the series

$$-.707, -.408, -.289, -.224, -.183, -.154, -.134, -.118, \ldots -1/\sqrt{(k+1)k}, \ldots .$$

3. Consider the $(n - 1) \times (n - 1)$ matrix resulting from the omission of the first row and the last column. The diagonal elements of this matrix, from lower right to upper left, are the first $(n - 1)$ terms of the series

$$.707, .816, .866, .894, .913, .926, .935, .943, \ldots \sqrt{k/(k+1)}, \ldots .$$

4. Each element below the diagonal of this $(n - 1) \times (n - 1)$ matrix is zero.

5. In each row, the remaining elements are each equal to the value of the last element of that row of the $n \times n$ matrix. Note that these elements are all negative.

A necessary and sufficient condition that a matrix transformation be orthogonal*† is that

$$\sum_{p=1}^{n} a^2_{ip} = 1,$$

and

$$\sum_{p=1}^{n} a_{ip} a_{jp} = 0, \qquad i \neq j, \quad \begin{matrix} i = 1, 2, \cdots n \\ j = 1, 2, \cdots n. \end{matrix}$$

For the i^{th} row, the value of k in either of the above series is given by $(n - i + 1)$, $i > 1$. The first element in the i^{th} row, other than zero, is an element of the $(n - 1) \times (n - 1)$ matrix and its value is $\sqrt{(n - i + 1)/(n - i + 2)}$. Each of the remaining $(n - i + 1)$ elements of the i^{th} row has the value

$$-1/\sqrt{(n - i + 2)(n - i + 1)}.$$

Then

$$\sum_{p=1}^{n} a^2_{ip} = \frac{n-i+1}{n-i+2} + (n-i+1) \frac{1}{(n-i+2)(n-i+1)} = 1.$$

The sum of the squares of the elements of the first row is also unity since each element has the value $1/\sqrt{n}$. The first part of the condition is then satisfied.

If we take $j > i$, the only elements of the i^{th} row not multiplied by zero in carrying out the second part of the condition are the last $(n - j + 2)$ elements. These elements are identical and have the value $-1/\sqrt{(n - i + 2)(n - i + 1)}$ for $i > 1$, and $1/\sqrt{n}$ for $i = 1$.

* Muir, Thomas and Metzler, William H., *A Treatise on the Theory of Determinants*, Menasha, Wisconsin: George Banta Publishing Company, 1929, p. 566.

† The condition may be stated with all subscript pairs interchanged.

In either case there is a common factor which we may designate as f_i. We then have, for $i \neq j$,

$$\sum_{p=1}^{n} a_{ip}a_{jp} = f_i\left[\sqrt{\frac{n-j+1}{n-j+2}} - (n-j+1)\sqrt{1/(n-j+2)(n-j+1)}\right] = 0.$$

The second part of the condition for orthogonality is then satisfied. Moreover, since the factor which vanishes is the sum of the elements of the j^{th} row, $j > 1$, the sum of the columns as vectors is a vector collinear with the first centroid.

A table of matrices, set up in accordance with the above procedure, is given for computational convenience. Alternative matrices are included for the cases $n = 4$ and $n = 8$. In each of these two cases, the absolute value of each element is $1/\sqrt{n}$. By inspection it can be seen that they satisfy the imposed conditions. One might draw attention to the fact that the mean of the entries in the first row of the final rotation matrix in the study "Primary Mental Abilities"* is .285, and the standard error is .053. The value of $1/\sqrt{n}$, where $n = 12$, is .289.

It may be that one† of the final reference vectors can be approximately specified from a knowledge of the factorial composition of one or more of the tests. A previous study or other considerations may be the source of this knowledge. The following suggestion permits a most advantageous use of this information.

Let the column matrix $|a_{ph}|$ represent the known vector and let M be the given centroid transformation matrix. Take the product given by the expression $\sum_{p=2}^{n} |a_{ph}a_{pj}|$ for each j, and let q be the value of j for which the expression has the greatest value.‡ Now change the signs§ of the rows of M so that the column q has the same arrangement of signs as the column of $|a_{ph}|$. Then

$$|a_{ph}|'|a_{pq}| = \sum_{p=1}^{n} |a_{ph}a_{pj}|.$$

The vector represented by column q is sufficiently close to that of $|a_{ph}|$ so that the new matrix may be directly applied, since usually the

* Thurstone, L. L., Primary Mental Abilities. 1938, Psychometric Monographs No. 1.

† The following may be generalized, approximately, for more than one vector.

‡ Most of the columns can be eliminated by inspection.

§ One might interchange the rows and obtain a still larger value for the expression given above, especially for the case of more than one known vector. Note that the orthogonality of a matrix is not affected by interchanging rows or by changing signs of rows; this fact is evident from the condition for orthogonality.

error in the approximately known vector is fairly large. If it is desired, however, the values of the matrix $|a_{ph}|$ may replace the column q. In this case the resulting matrix transformation is not exactly a centroid rotation nor exactly orthogonal. Even in this latter case few if any of the first set of graphs would require oblique plotting. Because the configuration is either orthogonal, or approximately so, there is an excellent chance that the graphs will indicate definite rotations of some of the vectors with the known vector. It is even more probable that the approximately known vector will show a rotation leading toward its final stable position.

TABLE OF MATRICES

.577	.577	.577	.500	.500	.500	.500	.500	.500	.500	.500	.500
.816	—.408	—.408	.866	—.289	—.289	—.289	.500	.500	—.500	—.500	
.000	.707	—.707	.000	.816	—.408	—.408	.500	—.500	.500	—.500	
			.000	.000	.707	—.707	.500	—.500	—.500	.500	

.477	.477	.477	.477	.477	.408	.408	.408	.408	.408	.408	
.894	—.224	—.224	—.224	—.224	.913	—.183	—.183	—.183	—.183	—.183	
.000	.866	—.289	—.289	—.289	.000	.894	—.224	—.224	—.224	—.224	
.000	.000	.816	—.408	—.408	.000	.000	.866	—.289	—.289	—.289	
.000	.000	.000	.707	—.707	.000	.000	.000	.816	—.408	—.408	
					.000	.000	.000	.000	.707	—.707	

.378	.378	.378	.378	.378	.378	.378	
.926	—.154	—.154	—.154	—.154	—.154	—.154	
.000	.913	—.183	—.183	—.183	—.183	—.183	
.000	.000	.894	—.224	—.224	—.224	—.224	
.000	.000	.000	.866	—.289	—.289	—.289	
.000	.000	.000	.000	.816	—.408	—.408	
.000	.000	.000	.000	.000	.707	—.707	

.354	.354	.354	.354	.354	.354	.354	.354
.935	—.134	—.134	—.134	—.134	—.134	—.134	—.134
.000	.926	—.154	—.154	—.154	—.154	—.154	—.154
.000	.000	.913	—.183	—.183	—.183	—.183	—.183
.000	.000	.000	.894	—.224	—.224	—.224	—.224
.000	.000	.000	.000	.866	—.289	—.289	—.289
.000	.000	.000	.000	.000	.816	—.408	—.408
.000	.000	.000	.000	.000	.000	.707	—.707

.354	.354	.354	.354	.354	.354	.354	.354
.354	—.354	.354	.354	—.354	—.354	—.354	.354
.354	.354	.354	—.354	—.354	.354	—.354	—.354
.354	.354	—.354	—.354	.354	—.354	—.354	.354
.354	—.354	—.354	.354	.354	.354	—.354	—.354
.354	—.354	.354	—.354	.354	—.354	.354	—.354
.354	—.354	—.354	—.354	—.354	.354	.354	.354
.354	.354	—.354	.354	—.354	—.354	.354	—.354

.333	.333	.333	.333	.333	.333	.333	.333	.333
.943	—.118	—.118	—.118	—.118	—.118	—.118	—.118	—.118
.000	.935	—.134	—.134	—.134	—.134	—.134	—.134	—.134
.000	.000	.926	—.154	—.154	—.154	—.154	—.154	—.154
.000	.000	.000	.913	—.183	—.183	—.183	—.183	—.183
.000	.000	.000	.000	.894	—.224	—.224	—.224	—.224
.000	.000	.000	.000	.000	.866	—.289	—.289	—.289
.000	.000	.000	.000	.000	.000	.816	—.408	—.408
.000	.000	.000	.000	.000	.000	.000	.707	—.707

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PSYCHOMETRIKA—VOL. 3. NO. 4
DECEMBER, 1938

A THEORY OF LEARNING AND TRANSFER: II

HAROLD GULLIKSEN AND DAEL L. WOLFLE
The University of Chicago

The theory of discrimination learning developed in Part I* of this paper is extended to more complex learning situations such as those used for differentiating relative from absolute habits. The greater difficulty of learning an absolute discrimination than of learning a relative discrimination is predicted. The theory is also extended to cover cases of transfer. It predicts that under the ordinary training conditions transfer to new stimulus situations will be on a relative basis, and it predicts the type of transfer to be expected under other training conditions. A number of additional established facts of learning and transfer can be deduced from the theory. Methods of testing the adequacy of the theory by new and more crucial experiments are suggested.

FOUR-CONFIGURATION PROBLEM

The line of reasoning employed in the analysis of situations involving two configurations can be extended to cases in which four configurations are used in the training. Four configurations, for example, must be used in establishing an absolute brightness discrimination and may be used in several other types of experiments. We will give several illustrations of the diverse experiments which are suggested by the present approach. Most of them, to our knowledge, have never been performed. In order to make the problems concrete, the illustrations will be made in terms of brightness discrimination. Consider the four configurations shown in figure 6, and diagramed as points 1, 2, 3, and 4 in figure 7.

Six different experiments are possible with different pairings of these four configurations. Two involve relative discrimination; two, absolute discrimination; and two, discrimination of total brightness.

Relative discrimination:

A. If a jump to the left is demanded in response to configurations 1 and 3, and a jump to the right in response to configurations 2 and 4, the experiment would usually be described as training the animal to respond to the relationship "brighter than."

B. If the responses demanded in *A* are reversed for all four con-

*Gulliksen, Harold and Wolfle, Dael L. "A Theory of Learning and Transfer: I," *Psychometrika*, **3** 1938, 127-149.

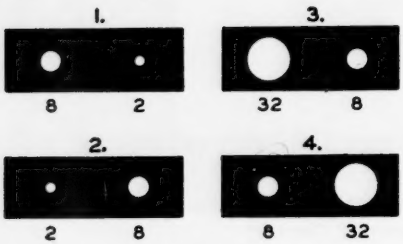


FIGURE 6
Four configurations to be used in training for relative discrimination, absolute discrimination, or discrimination of total brightness.
The numbers beneath the circles represent the relative brightness of the stimulus cards.

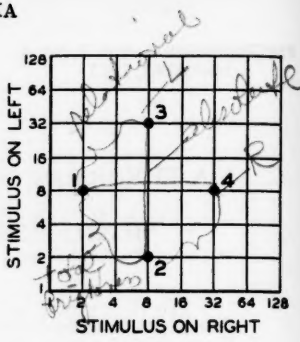


FIGURE 7
Plot of four configurations. By different pairings of these four configurations, training may be given on any of the six problems described on pages 225-227.

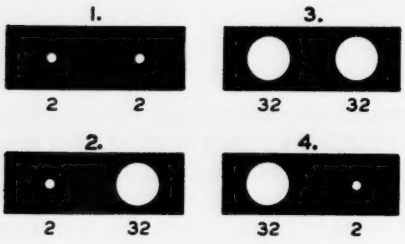


FIGURE 8
Four configurations to be used in training for the six problems described on page 227.

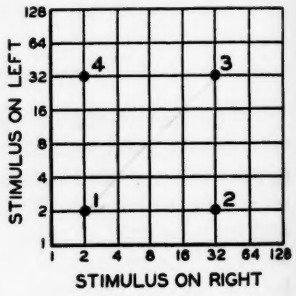


FIGURE 9
By different pairings of these four configurations, training may be given on any of the six problems described on page 227.

figurations, it would be said that the animal was being trained to respond to the relationship "dimmer than."

Absolute discrimination:

C. If a jump to the left is demanded in response to configurations 1, and 4, and a jump to the right in response to configurations 2 and 3, the animal is being trained to respond positively to an absolute brightness of 8 units.

D. If the responses demanded in *C* are reversed for all four configurations, the animal is being trained to avoid an absolute brightness of 8 units.

Discrimination of total brightness:

E. If a jump to the left is demanded in response to configurations 1 and 2, and a jump to the right in response to configurations 3 and 4, the experiment may be described by saying that when the total brightness is low (10 units) the animal is required to jump to the left. When the total brightness is high (40 units) it is required to jump to the right. This experiment might also be described by saying that the animal is trained to go to the left whenever one of the lights is very dim and to the right whenever one of the lights is very bright.

F. As formerly, the responses demanded in *E* may be reversed for all four configurations.

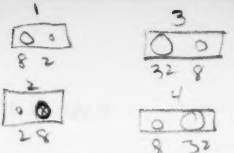
We do not know that the last three experiments have ever been performed.

It is possible to make a similar analysis involving any set of four configurations. If the four configurations illustrated in figure 8 and diagramed in figure 9 are used, there are, again, six different experiments which may be performed. If, for example, a jump to the left is demanded in response to configurations 1 and 3, and to the right in response to configurations 2 and 4, we are training the subject to go to the left when the two stimuli are the same and to the right when they are different. As before, this training may be reversed. The other four experiments with this set of configurations are all essentially alike and we will give only one illustration. If we train the subject to jump to the right in response to configurations 1 and 2, and to the left in response to configurations 3 and 4, we may describe this experiment by saying that we require the subject to attend to the *left* stimulus of each configuration. If that stimulus is dim he goes to the right; if it is bright he goes to the left. The three other experiments are analogous to this one.

With each possible set of four configurations six different experiments could, in general, be performed. Of all experiments possible with four configurations, only two types have been reported in the literature — training the animal to discriminate relative brightness and training it to discriminate absolute brightness. The theoretical formulation to be given should apply to any possible experiment with four configurations and hence may be tested not only in the conventional relative and absolute discrimination problems but in a large number of new experimental situations as well.

Definitions

The theoretical formulation may now be extended to include the four-configuration problem.



Using odd-numbered subscripts, 1 and 3, to designate configurations for which one response is required and even-numbered subscripts to indicate configurations for which the other response is required, by analogy with equations (II-7) and (II-8) we may write for configuration 1:

$$\frac{e_1}{s_1} = \frac{du_1}{dw_1} = \frac{g_1 - cu_1 - c_{13}u_3 + k_{12}w_2 + k_{14}w_4}{h_1 + kw_1 + k_{13}w_3 - c_{12}u_2 - c_{14}u_4}. \qquad \text{(IV-1)*}$$

The different terms have the same meanings as in equation (II-7). k_{1j} and e_{1j} ($j = 2$, 3, or 4) represent the influence of reward or punishment of a response to configuration j upon the strength of the tendency to give the same response to configuration 1.

The differential equations for configurations 2, 3, and 4 may be obtained from (IV-1) by the proper interchange of subscripts. Interchanging subscripts 1 with 2 and 3 with 4 gives the differential equation for configuration 2. Interchanging subscripts 1 with 3 and 2 with 4 gives the differential equation for configuration 3. Interchanging subscripts 1 with 4 and 2 with 3 gives the differential equation for configuration 4.

We will solve these equations for the special case in which the four configurations are so chosen that, when plotted as in figure 7, they lie on the corners of a rectangle. In this case:

The distance 1-2 equals the distance 3-4.

The distance 1-3 equals the distance 2-4.

The distance 1-4 equals the distance 2-3.

From these equalities the learning parameters may be redefined as follows:

$$c_{12} = c_{34} = c_1\,, \qquad \text{(IV-2)}$$

$$k_{12} = k_{34} = k_1\,, \qquad \text{(IV-3)}$$

$$c_{13} = c_{24} = c_2\,, \qquad \text{(IV-4)}$$

$$k_{13} = k_{24} = k_2\,, \qquad \text{(IV-5)}$$

$$c_{14} = c_{23} = c_3\,, \qquad \text{(IV-6)}$$

$$k_{14} = k_{23} = k_3\,. \qquad \text{(IV-7)}$$

The subscript 1 now indicates the distance from any configuration to the closer, and the subscript 3 indicates the distance to the farther, of the two configurations to which the opposite response is required. The subscript 2 indicates the distance to the other configuration to

* The number of each equation dealing with the four-configuration case will be preceded by IV.

which the same response is required.

It is assumed that $g_1 = g_2 = g_3 = g_4$,

and that $h_1 = h_2 = h_3 = h_4$.

Substituting equations (IV-2) to (IV-7) inclusive in (IV-1) we have — with changes of subscripts indicated above — four equations.

Making the assumption that (see equation II-10)

$$w_1 = w_2 = w_3 = w_4, \tag{IV-8}$$

we may by a process similar to that outlined in conjunction with equation (II-13) show that

$$u_1 = u_2 = u_3 = u_4 \tag{IV-9}$$

under the same restrictions as in the previous case.

Derivation

The identities of equations (IV-8) and (IV-9) make it necessary to solve only one equation, namely

$$\frac{du}{dw} = \frac{g - (c + c_2)u + (k_1 + k_3)w}{h + (k + k_2)w - (c_1 + c_3)u}. \tag{IV-10}$$

By setting

$$(c + c_2) = c, \tag{IV-11}$$

$$(k + k_2) = k, \tag{IV-12}$$

$$(c_1 + c_3) = c', \text{ and} \tag{IV-13}$$

$$(k_1 + k_3) = k', \tag{IV-14}$$

it can be seen that equation (IV-10) is formally equivalent to equation (II-15) in the two-configuration case.

Substituting equations (IV-11) to (IV-14) inclusive in equations (II-33), (II-38), (II-39) or (II-40) gives the corresponding solutions for equation (IV-10). Similarly the equations for the asymptotes, their intersection, the tangent or secant of the angle between them, and the semi-major axis of the equation for the four-configuration problem may be written, by analogy, from the corresponding equations of the two-configuration problem.

To make a direct and simple comparison of the two- and four-configuration cases we will use only equation (II-40) of the former derivation in which $g = h$ and $c = k$. This equation is

$$u^2 + w^2 - \frac{2kuw}{k'} - \frac{2gu}{k'} + \frac{2gw}{k'} = 0. \tag{II-40}$$

Making the substitutions indicated in equations (IV-11) to (IV-14) and setting $c_i = k_i$ gives the corresponding four-configuration form as follows:

$$u^2 + w^2 - 2\frac{k + k_2}{k_1 + k_3}uw - \frac{2g}{k_1 + k_3}u + \frac{2g}{k_1 + k_3}w = 0 . \tag{IV-16}$$

Properties of the Learning Curve for the Four-Configuration Problem

The slope of the asymptote of the learning curve for the four-configuration problem is:

$$\frac{k + k_2 - \sqrt{(k + k_2)^2 - (k_1 + k_3)^2}}{k_1 + k_3} .$$

If a new variable, x, be defined as

$$x = \frac{k + k_2}{k_1 + k_3} \tag{IV-17}$$

the slope of the asymptote may be rewritten as

$$x - \sqrt{x^2 - 1} . \tag{IV-18}$$

The derivative of this expression is negative for values of x from 1 to ∞, which shows that as x increases above values of 1, the slope of the asymptote decreases. If $x < 1$, the radical becomes imaginary, and if $x = 1$, the slope of the asymptote equals unity. Consequently, if learning is to occur, the condition

$$\frac{k + k_2}{k_1 + k_3} > 1 \tag{IV-19}$$

must be satisfied.

Application to Experimental Data

For any given set of four configurations, k will always retain the same value for a given subject regardless of whether he is being trained on a relative, absolute, or total size discrimination. The value of x [Equation (IV-17)] will, then, depend upon the relative sizes of k_1, k_2, and k_3, which will vary with the particular problem.

In figure 10, the four configurations are plotted. The symbols p, m and n represent the amount of transfer from configurations 2, 3, and 4 to configuration 1. Since transfer is an inverse function of distance it is obvious that:

TABLE 2

Comparison of properties of the asymptotes of the equations for the two-configuration and four-configuration problems. The formulae are here given in their simplest forms.

Properties	Two-configuration Problem	Four-configuration Problem
Equation of Asymptotes	$u = \left[\frac{k \pm \sqrt{k^2 - k'^2}}{k'}\right] w + \frac{g}{k'}\left[1 \pm \sqrt{\frac{k-k'}{k+k'}}\right]$	$u = \left[\frac{k + k_2 \pm \sqrt{(k+k_2)^2 - (k_1+k_3)^2}}{k_1 + k_3}\right] w + \frac{g}{k_1 + k_3}\left[1 \pm \sqrt{\frac{(k+k_2)-(k_1+k_3)}{(k+k_2)+(k_1+k_3)}}\right]$
Intersection of Asymptotes	$u = \frac{g}{k + k'}$ $w = -\frac{g}{k + k'}$	$u = \frac{g}{k + k_1 + k_2 + k_3}$ $w = -\frac{g}{k + k_1 + k_2 + k_3}$
Secant of the angle between asymptotes	$\frac{k}{k'}$	$\frac{k + k_2}{k_1 + k_3}$

$k > m > n$, and

$k > p > n$.

In the case of relative discrimination one response is given to configurations 1 and 3 and the other response to configurations 2 and 4. In this case $k_1 = p$, $k_2 = m$, and $k_3 = n$. The value of x [Equation (IV-17)] will then be

$$x = \frac{k+m}{p+n}. \tag{IV-20}$$

In the case of absolute discrimination one response is given to configurations 1 and 4 and the other response is given to configurations 2 and 3. In this case $k_1 = p$, $k_2 = n$, $k_3 = m$. The value of x will then be

$$x = \frac{k+n}{p+m}. \tag{IV-21}$$

(1) It may be deduced that *the ultimate level of accuracy* (as represented by the slope of the asymptote) *attainable in an absolute discrimination will be lower than that attainable in a relative discrimination involving the same four configurations.*

The level of accuracy attainable will be less when the asymptote of the learning curve has a greater slope. From equation (IV-18), it is clear that the slope of the asymptote increases with a decrease in the value of x. Since $m > n$, x in equation (IV-20) is greater than x in equation (IV-21), *i.e.*,

$$\frac{k+m}{p+n} > \frac{k+n}{p+m}.$$

If the slope of the asymptote is taken as a measure of the difficulty of the problem, the conclusion is reached that an absolute brightness discrimination is more difficult than the corresponding relative discrimination.

No work has been reported on the *ultimate* level of accuracy attainable under the two types of conditions. But if the two conditions differ in this respect, it is to be expected that more trials will be required to reach a given level of accuracy with an absolute than with a relative discrimination. This expectation is borne out by the experiments of Warden and Rowley (30)* using the ring dove. Warden and Winslow (31) using the ring dove, and Wolfle (33) using the white rat.

* The references given in this paper will be found in *Psychometrika*, Vol. 3, pages 148-149.

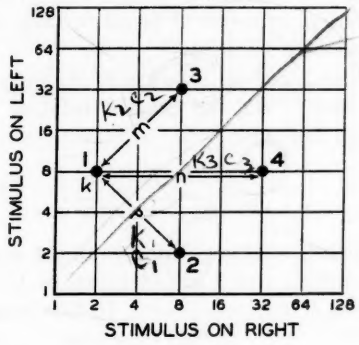


FIGURE 10

Plot of four configurations. The values of k, p, m, and n represent the amounts of transfer from configurations 1, 2, 3, and 4 to configuration 1.

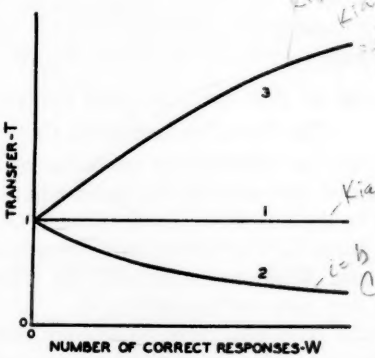


FIGURE 11

The relationship between consistency of transfer and previous number of correct responses to the training configuration for three different test configurations.

(2) If an animal is trained to go to the left (or right) in response to configurations 1 and 3 of figure 7 and to the right (or left) in response to configurations 2 and 4, and if the distances from 1 to 2 and from 3 to 4 are held constant, it may be deduced that *the ultimate level of accuracy*, as measured by the slope of the asymptote, *is the*

same when configurations 1 and 3 (and 2 and 4) are far enough apart to be essentially independent as it is when they are so close together that they are practically identical.

If configurations 3 and 4, in figure 7, are very far from configurations 1 and 2, k_2 and k_3 (in the case of relative discrimination) are essentially zero. In this case

$$x = \frac{k + k_2}{k_1 + k_3} = \frac{k}{k_1}.$$

If, on the other hand, configurations 3 and 4 are gradually brought closer to 1 and 2 until 1 and 3 are identical and 2 and 4 are identical, k_2 approaches k and k_3 approaches k_1. In this case the four configurations have reduced to two. In this case also

$$x = \frac{k + k_2}{k_1 + k_3} = \frac{2k}{2k_1} = \frac{k}{k_1}.$$

The values of x and the slopes of the asymptote must be the same for the two extreme cases. The optimum of ease or difficulty between these limits will depend upon the function relating k to the psychological distance between the two configurations involved. "Ease" or "difficulty" refers here to the ultimate level of accuracy attained regardless of the number of trials or errors required to reach this level.

Transfer of Response to New Configurations

The Two-Configuration Case

The theory may be extended to problems of transfer. If an animal has been trained to jump to the right when presented with configuration a and to the left when confronted with configuration b, we may designate a jump to the right as an a-response and a jump to the left as a b-response. What response will the animal now give if it is confronted with any new configuration, i?

Let a_i = the initial strength, prior to any training on a and b, of the tendency to give the a-response to configuration i,

and b_i = the initial strength, prior to any training on a and b, of the tendency to give the b-response to configuration i.

If t_a denotes the cumulative count of a-responses given to i, and t_b denotes the cumulative count of b-responses given to i, then, following the formulation of the two-configuration case, we may write

$$\frac{dt_a}{dt_b} = \frac{a_i + k_{ia}w_a - c_{ib}u_b}{b_i - c_{ia}u_a + k_{ib}w_b}. \qquad \text{(T-1)*}$$

* The number of each equation dealing with the problem of Transfer will be preceded by T.

Here the u's and w's represent the cumulative count of incorrect and correct responses given to configurations a and b. The coefficients of these variables are functions of the animal's learning ability and the distance from i to a and to b. The initial differential equation differs only in its subscripts from that for the two-configuration problem.

If $\frac{dt_a}{dt_b} = 1$, then the tendency to give the a-response will be equal to the tendencey to give the b-response, and there will be no consistent tendency for either the a-response or the b-response to predominate. If $\frac{dt_a}{dt_b} > 1$, then the a-response is more likely to be given. If $\frac{dt_a}{dt_b} < 1$, the b-response is more likely to be given. Since $\frac{dt_a}{dt_b}$ represents the amount and direction of transfer to be expected, we may substitute T for $\frac{dt_a}{dt_b}$.

If the initial strength a_i and b_i as well as the coefficients of the u's and w's were known from previous experiments it would be easy to substitute the cumulative correct and incorrect responses to both a and b at any stage of learning and predict the relative strengths of the a-response and b-response tendency for any configuration i.

As with the learning problem, we may simplify the treatment by considering only the case where

$$a_i = b_i ,$$

$$u_a = u_b , \text{ and}$$

$$w_a = w_b .$$

If, in addition, we assume that $c = k$, the transfer effects will show up more clearly. With these assumptions (T-1) becomes

$$T = \frac{a_i + k_{ia}w - k_{ib}u}{a_i - k_{ia}u + k_{ib}w} . \qquad \text{(T-2)}$$

From equation (II-41) we can see that $u = \varphi(w)$ so that substituting (II-41) in (T-2) will give

$$T = \frac{(a_i - k_{ib}\frac{g}{k'}) + (k_{ia} - k_{ib}\frac{k}{k'})w + k_{ib}\sqrt{F(w)}}{(a_i - k_{ia}\frac{g}{k'}) + (k_{ib} - k_{ia}\frac{k}{k'})w + k_{ia}\sqrt{F(w)}} , \qquad \text{(T-3)}$$

where

$$F(w) = \left[\left(\frac{k}{k'}\right)^2 - 1\right]w^2 + \frac{2g}{k'}\left(\frac{k}{k'} - 1\right)w + \left(\frac{g}{k'}\right)^2 .$$

Properties of This Equation

(1) It can be seen from equation (T-2) that if $k_{ia} = k_{ib}$, then T is unity for all values of w (See figure 11, curve 1.).

(2) As k_{ia} decreases and k_{ib} increases, T becomes smaller. When $i = b$, $k_{ib} = k$, which is its maximum value, and $k_{ia} = k'$. For this point, the transfer values may be designated by T_b. Substituting these values in equation (T-3) and setting $a_i = g$, gives

$$T_b = \frac{k}{k'} - \frac{F'(w)}{2\sqrt{F(w)}}. \tag{T-4}$$

If $w = 0$, T is unity and T_b is unity, and as

$$w \to \infty, T_b \to \frac{k}{k'} - \sqrt{\frac{k^2}{k'^2} - 1}.$$

The first derivative of (T-4) is negative, so the curve relating transfer to previous numbers of correct responses is as shown in figure 11, curve 2.

(3) When $i = a$, we will have large T-values, designated by T_a, for any given w. This transfer curve is found by setting $k_{ib} = k'$ and $k_{ia} = k$. It can be seen from equation (T-2) that an interchange of k_{ia} and k_{ib} gives a new value of T which is the reciprocal of the former one, so that the curve for T_a is equal to $1/T_b$. This curve is shown as curve 3 of figure 11.

The derivative of (T-3) may be written

$$\frac{dT}{dw} = \frac{\left(1 - \frac{k_{ib}}{k_{ia}}\right)\left[\left(\frac{k}{k'} - 1\right)Mw + \frac{g}{k'}\left(M + 2\frac{a_i}{k_{ia}}\right) - M\sqrt{F(w)}\right]}{\sqrt{F(w)}\left[\left(\frac{a_i}{k_{ia}} - \frac{g}{k'}\right) + \left(\frac{k_{ib}}{k_{ia}} - \frac{k}{k'}\right)w + \sqrt{F(w)}\right]^2}, \tag{T-5}$$

where $F(w)$ has the same meaning as in equation (T-4) and

$$M = \frac{g}{k'}\left(1 + \frac{k_{ib}}{k_{ia}}\right) - \frac{a_i}{k_{ia}}\left(\frac{k}{k'} + 1\right).$$

It can be seen from equation (T-5) that $\frac{dT}{dw}$ will become zero under three conditions.

(1) Dividing both numerator and denominator by w, it can be seen that as $w \to \infty$, $\frac{dT}{dw} \to 0$. Thus, as the amount of practice becomes very large, T approaches a stable value.

(2) If $k_{ia} = k_{ib}$, $\frac{dT}{dw}$ will equal zero for all values of w, as shown in curve 1 of figure 11.

(3) Any value of w for which the bracketed expression in the numerator of (T-5) vanishes will make $\frac{dT}{dw}$ equal to zero. If the second derivative is negative or positive, such values of w will represent the amount of training for which transfer is, respectively, either a maximum or a minimum.

Applications to Experimental Data

(1) It may be deduced that: *If the animal has been trained to go to the larger (or smaller) of two stimuli regardless of whether this stimulus is on the right or left side, it will respond on a relational basis in transfer tests.*

Assume that the animal was trained to go to the left for configuration A and to the right for configuration B, that is, to the larger of the two stimuli, as diagramed in figure 12. The tendency to jump to the left will be equal to the tendency to jump to the right for all configurations for which T equals unity. These will be configurations for which k_{ia} equals k_{ib}, i.e., those which are equidistant from A and B. These configurations will be on the diagonal line shown in figure 12. Since T equals unity for all configurations on the line, it may be called the line of indifferent transfer, or the "indifference line." All configurations on this line are composed of two stimuli which are equal in size. For configurations lying above and to the left of this line, $k_{ib} < k_{ia}$, since these configurations are nearer to A than to B. It can readily be seen from equation (T-2) that in this case $T > 1$, that is, the tendency to give the A-response is stronger. Since the A-response in this illustration is a jump to the left, this means that the animal will jump to the left for all configurations above and to the left of the line in figure 12. Since these are configurations with the larger stimulus on the left the animal will also be jumping to the "larger of the two" stimuli. For configurations below and to the right of the line, the B-response will be stronger. This response is a jump to the right in the present illustration and these configurations are ones in which the stimulus on the right side is larger. Thus it can be seen that jumping to the left for configurations more like A and to the right for configurations more like B may also be described as jumping to the "larger of the two" in every case. We have then the familiar fact of "transposition" or "transfer on a relational basis" (3, 6, 7, 8, 14, 22, 26) predicted from a theory which did not assume the existence of relational response tendencies.

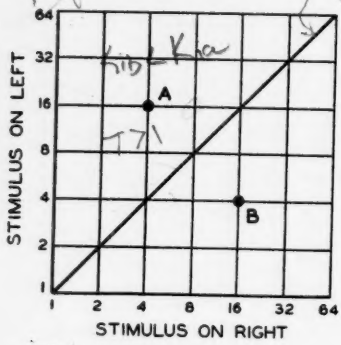


FIGURE 12

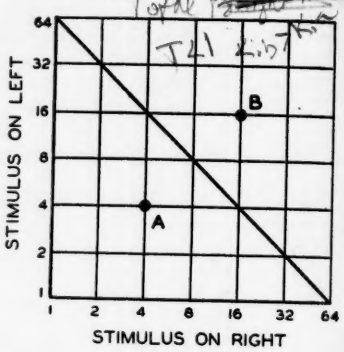


FIGURE 13

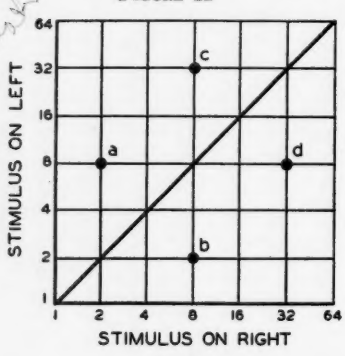


FIGURE 14

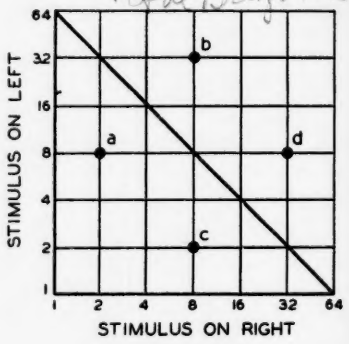


FIGURE 15

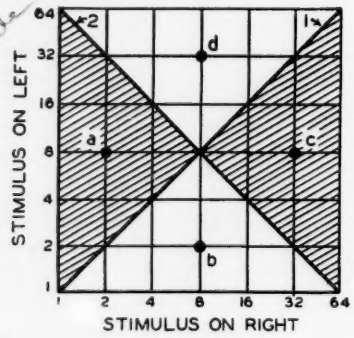


FIGURE 16

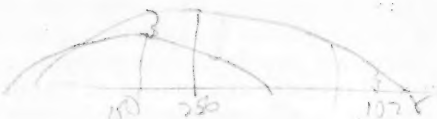
Figure 12. If the animal is trained to jump to the left in response to configuration A and to the right in response to configuration B, it will jump to the left in response to any configuration above and to the left of the diagonal line and to the right in response to any configuration below and to the right of the diagonal line. Responses to configurations on the diagonal line will be made on a chance basis.

Figure 13. If the animal is trained to jump to the left in response to configuration A and to the right in response to configuration B, it will respond to test configurations by jumping to the right in response to all configurations above and to the right of the diagonal line and to the left in response to all configurations below and to the left of the diagonal line. Responses to configurations on the diagonal line will be made on a chance basis.

Figure 14. Transfer in a four-configuration relative discrimination problem. If the animal has been trained to jump to the left, in response to configurations *a* and *c*, and to the right in response to configurations *b* and *d*, it will respond on test trials by jumping to the left to any configuration above and to the left of the diagonal line and to the right in response to any configuration below and to the right of the diagonal line. Responses to configurations on the diagonal line will be made on a chance basis.

Figure 15. Transfer in a four-configuration total brightness discrimination problem.
If the animal has been trained to jump to the left in response to configurations *a* and *c* and to the right in response to configurations *b* and *d*, it will respond on test trials by jumping to the left to any configuration below and to the left of the diagonal line and to the right in response to any configuration above and to the right of the diagonal line. Responses to configurations on the diagonal line will be made on a chance basis.

Figure 16. Transfer in a four-configuration absolute discrimination problem. If the animal has been trained to jump to the left in response to configurations *a* and *c* and to the right in response to configurations *b* and *d*, it will respond on test trials by jumping to the left in response to any configuration in the shaded areas and by jumping to the right in response to any configuration in the unshaded areas. Responses to configurations on either of the two diagonal lines will be made on a chance basis.



Spence (26) has presented some evidence which indicates that under certain special conditions the animal may reverse the usual tendency to respond on a relational basis in transposition tests.

Lashley (19), in a discussion of the basis of response of the rat, and Hebb (6, 7), in an experiment designed to test the rat's innate visual organization, both reach the definite conclusion that the animal's response is based on relational characteristics of the two stimuli being discriminated. In both cases, the only evidence cited is the fact that in transfer tests the animal responds on a relational basis. *Transfer* on a relational basis is, in these articles, interpreted as evidence that *learning* was on a relational basis. It has now been demonstrated that relative transfer can be predicted from a theory based on a directional definition of the response. Since this prediction is possible it is not legitimate to use the fact of relational transfer to support the assumption that the animal's original learning was relational in nature.

(2) It may be deduced that: *If the animal has been trained to go to one side in response to configurations containing large stimuli, and to the other side in response to configurations containing small stimuli, it will respond on the basis of total size of stimuli in transfer tests.*

Assume that the animal was trained to go to the left for configuration A and to the right for configuration B, as diagramed in figure 13. Assume further that the logarithmic assumption, on the basis of which the configurations in figure 13 were plotted, is correct. Then by a line of reasoning analogous to that followed above, it can be seen that the configurations for which the tendency to jump to the left is equal to the tendency to jump to the right will lie on the "indifference line" shown in figure 13. For configurations lying above this line the B-response will be stronger, and for configurations lying below this line the A-response will be stronger. In other words, for configurations containing large stimuli the B-response will be given, and for those containing small stimuli the A-response will be given.

Experimental determination of where the indifference line runs in this case would give some important properties of the configurational field involved. For example, if the logarithmic scale suggested to conform with Weber's law is correct, the indifference line should lie where the product of the two brightness values involved in the configuration is equal to the geometric mean of the two brightness values used in training. Variations from this line would indicate the type of modification of the logarithmic principle that should be made.

(3) There are two deductions concerning the relationship between accuracy of transfer and amount of training:

(a) It may be deduced that: *For certain values of the parameters*

the accuracy of transfer to any given configuration will increase with an increase in training. That is, if the configuration i is nearer to B than to A, the relative strength of the tendency to give the B-response to configuration i will increase with increased training on the A-B discrimination. Conversely, if the configuration is nearer to A than to B, the relative strength of the tendency to give the A-response to that configuration will increase with increased training on the A-B discrimination.

That this deduction follows from the properties of equation (T-3) can be seen from the curves in figure 11 and from equation (T-4). Since the accuracy attained increases with increased training, a corollary of the foregoing deduction would be that: *The accuracy of transfer to a given configuration will increase with an increase in the accuracy attained in the original discrimination.* Gulliksen (5) has shown that the consistency of transfer by white rats in a size-brightness discrimination problem increases with an increase in the accuracy attained before giving the transfer tests. Hovland (10), working with a conditioned galvanic response to tonal stimuli, found that with increased training there was a gradual increase in the absolute magnitude of the galvanic response to tonal stimuli different from the training one.

(b) It may also be deduced that: *For other values of the parameters of the equation, the accuracy of transfer to any given configuration will first increase, and then decrease with increase in training.*

That this deduction follows can be seen from equation (T-5), the derivative of the transfer equation (T-3).

Hovland (10) really found a steady increase; plotting the ratio of generalized to conditioned responses gave his figure 2, an apparent decrease in generalization.

In the case of both Gulliksen's and Hovland's data it would be necessary to know the values of the various parameters involved in the transfer equation (T-3), in order to determine whether or not these experiments verify the theory.

By combining learning and transfer problems it is possible to get additional checks on the theory. For example, if a given configuration i is equidistant from a and b as shown by transfer experiments, then learning experiments should show that a problem requiring a discrimination between i and a is equal in difficulty to one requiring a discrimination between i and b. Difficulty, as used here, means the ultimate level of accuracy attained. That is, for animals with the same learning ability k, the secant of the angle between the asymptotes $(\frac{k}{k'})$ should be the same for both experiments.

Similarly, if i is nearer to a than to b as shown by transfer experiments, then, in learning experiments the discrimination between i and a should be less accurate than the discrimination between i and b. For animals of the same learning ability, the ratio of the secants of the angles between the two asymptotes $[\frac{k}{k_{ib}} \div \frac{k}{k_{ia}}]$ should equal $\frac{k_{ia}}{k_{ib}}$. See equation (T-2).

The Four-Configuration Case

Extending formula (T-1) to the four-configuration case is relatively easy.

$$\frac{dt_a}{dt_b} = \frac{a_i + k_{ia}w_a + k_{ic}w_c - c_{ib}u_b - c_{id}u_d}{b_i - c_{ia}u_a - c_{ic}u_c + k_{ib}w_b + k_{id}w_d}. \qquad \text{(T-6)}$$

The deductions regarding transfer will stand out more clearly if we assume

$w_a = w_b = w_c = w_d = w,$

$u_a = u_b = u_c = u_d = u,$

$a_1 = b_i,$ and

$k_i = c_i .$

In addition, substituting T for $\frac{dt_a}{dt_b}$ gives:

$$T = \frac{a_i + (k_{ia} + k_{ic})w - (k_{ib} + k_{id})u}{a_i - (k_{ia} + k_{ic})u + (k_{ib} + k_{id})w}. \qquad \text{(T-7)}$$

This equation has the same properties as equation (T-1) if the appropriate change in coefficients is made. The foregoing deductions hold in an analogous manner.

We will consider here only the indifference lines for the four-configuration case. It can be seen from equation (T-7) that $T = 1$ if

$$k_{ia} + k_{ic} = k_{ib} + k_{id} .$$

Let us draw the indifference lines for the three possible types of learning problems using four configurations.

1. *Relational Discrimination.*

In figure 14, the indifference line for the relational habit is shown. Since $k_{ia} = k_{ib}$ and $k_{ic} = k_{id}$ anywhere along the line, T equals unity.

This case is analogous to the corresponding two-configuration case already discussed. If the animal has been trained to make one response to configurations a and c and the other response to configurations b and d, transfer to new configurations will be relational in character.

2. *Total Brightness Discrimination.*

If one response is demanded to configurations a and c in figure 15, and the opposite response to configurations b and d, the training conditions demand a discrimination of total brightness. Anywhere along the line in figure 15

$$k_{ia} = k_{ib} \text{ and } k_{ic} = k_{id}.$$

Transfer to any configuration on this line will be equal, or, in other words, the discrimination will break down.

3. *Absolute Discrimination.*

If one response is demanded to configurations a and c in figure 16 and the other response is demanded to configurations b and d, an absolute discrimination habit is formed. Assuming that the conditions for learning the problem are satisfied, namely

$$\frac{k + k_{ac}}{k_{ab} + k_{ad}} > 1$$

(See Equation IV-19)

then $k_{ia} + k_{ic} > k_{ib} + k_{id}$ for configurations near a and c, and the a-response will be given. For configurations near b or d the reverse is true. Under these training conditions, transfer will be on an absolute basis.

With these training conditions, there will be two indifference lines or lines of equal transfer. These lines are labeled 1 and 2 in figure 16.

Along line 1, $k_{ia} = k_{ib}$ and $k_{ic} = k_{id}$.

Along line 2, $k_{ia} = k_{id}$ and $k_{ib} = k_{ic}$.

In either case $k_{ia} + k_{ic} = k_{ib} + k_{id}$ and the value of $T = 1$. The discrimination will break down if the animal is presented with configurations lying on either of these two lines.

SUMMARY AND DISCUSSION

We have selected the problem of visual discrimination by the white rat to illustrate the development of a rational learning theory which is applicable to any situation in which the learning subject is required to respond in one way to one class of configurations and in

another way to another class of configurations. In our example, we have defined the response as being a jump to the right or a jump to the left, and the stimulus as consisting of the entire configuration presented on a given trial.

In addition to these basic definitions, we have assumed that: (1) each rewarded response increases the tendency to repeat that response upon later presentations of the configuration to which it was made and also, but to a lesser extent, upon later presentations of other similar configurations; (2) each punished response decreases the tendency to repeat that response upon later presentations of the configuration to which it was made and also, but to a lesser extent, upon later presentations of other similar configurations; (3) the slope of the cumulative error curve is equal to the ratio of the strengths of the tendencies to respond incorrectly and correctly.

On the basis of these fundamental definitions and assumptions (and with the help of certain simplifying conditions described in the paper), a rational equation of the learning curve has been developed. This equation was shown to be a hyperbola passing through the origin. With appropriate changes in the parameters it was shown that this equation describes the course of learning when four configurations are used instead of two and that it predicts the nature of the transfer to new configurations.

A number of deductions may be made rigorously from the equation. We have presented the following ones:

(1) The difficulty of a problem, as measured by the maximum level of accuracy attainable, is inversely related to the distance separating the two configurations to be discriminated. Direct experimental work on learning is not available to verify this deduction. It is in agreement with a large body of psychophysical data.

(2) The ultimate level of accuracy attainable in an absolute discrimination will be lower than that attainable in a relative discrimination involving the same four configurations. This deduction is in agreement with experimental findings.

(3) The ultimate level of accuracy of discrimination in a four-configuration problem, as measured by the slope of the asymptote, is the same for the extreme cases of pairs of configurations far enough apart to be essentially independent and for pairs of configurations close enough together to be practically identical. No experimental evidence on this point has been reported.

(4) If the animal has been trained to go to the larger (or smaller) of two stimuli, it will respond on a relational basis in transfer tests. A large mass of experimental data reported by a number of workers is in agreement with this deduction.

(5) If the animal has been trained to go to one side when presented with two large stimuli, and to the other side when presented with two small stimuli, it will respond to test configurations on the basis of total size of stimuli. No experimental evidence on this point has been reported.

(6) Under some conditions, consistency of transfer to any given configuration will increase with each increase in the amount of training preceding the transfer test. Under other conditions, the consistency of transfer will reach a maximum with an optimal amount of training and will then decrease. The first part of this deduction is in agreement with experimental findings; there is no evidence on the second part of the deduction.

(7) Consistency of transfer will decrease as the distance between the training configurations and the test configurations increases. This deduction is in agreement with experimental findings.

(8) If training has established an absolute discrimination, transfer to new configurations will be on an absolute basis. No experimental evidence on this point has been reported.

The preceding deductions have been made with specific reference to size discrimination. As mentioned in the Introduction, similar deductions may be made for discrimination of other characteristics, such as hue, saturation, and brightness.

The experimental testing of these deductions provides the soundest method of testing the plausibility of the theory on which they were based. We have indicated that there is experimental evidence in qualitative agreement with some of the deductions. Some of the deductions are completely untested. All of them are untested quantitatively. When the stimulus configurations are accurately scaled, the theory leads to a number of quantitative deductions regarding the ease or difficulty of different problems, and the certainty of transfer to given new configurations. If no qualitative disagreement between theoretical deduction and experimental fact is found, it will become desirable to make these quantitative predictions and test them experimentally.

In developing the theory we have applied it to the problems of learning and transfer of visual discriminations by the white rat. The specific definitions and postulates used were made with this type of learning in mind. But the theory is more general than the illustration. By appropriate changes in the definitions and in some of the postulates, a similar development could be made for learning of any type of material to which the concept of scaling can be applied.

An equation has been presented for the two-configuration case and extended to the four-configuration case. It may be further ex-

tended to cover situations in which more than four configurations are involved — the n-configuration case.

Furthermore, by extension, the theory can be applied to more complex situations. Some experimenters have used a technique in which three or more stimuli were presented on each trial. To handle such situations we must assume that the response possibilities are no longer limited to a jump to the right and a jump to the left, but include other possible types of response, such as a jump to the middle. Corresponding to this increase in the number of response possibilities there is a probable increase in the dimensionality of the space in which the stimulus configurations are plotted. If, for example, three stimuli are presented on each trial, the configurations could be plotted in terms of three coordinates. To extend the formulation to handle form discrimination may involve a similar increase in dimensionality. The distance between configurations consisting of different forms could be measured, and this problem handled in essentially the same way as the size or brightness discrimination problem. The labor involved in handling the n-dimensional case would be greatly magnified, but otherwise the 2-dimensional case and the n-dimensional case are analogous.

Extension to the n-configurational case in n-dimensional space would allow treatment of some problems which are not ordinarily classed as discrimination learning. It may be pointed out as an example that concept formation or the learning of a generalization may be thought of as a discrimination problem involving a large number of configurations to which the same response is demanded. The learning by a child of the meaning of the concepts "bread," "spinach," and "orange" certainly demands that he be able to discriminate between the different stimulus configurations appropriate to these concepts and that he be able to give the same response to a number of similar configurations. In the examples chosen it may be predicted that the distances separating these configurations are fairly large. Learning, consequently, should be easier and incorrect transfer to new configurations less likely than where smaller distances are involved, as for example, in learning to discriminate between the situations in which conceptual responses of "duck," "chicken," and "goose" are appropriate.

Reports are occasionally made of positive transfer from one sense mode or one kind of stimulus to another sense mode or kind of stimulus. Thus, if an animal has been trained to go to the brighter of two lights, it may respond positively to the larger of two circles or the louder of two tones. Transfer of this type can be predicted theoretically in exactly the same manner as transfer under the more usual conditions. If the distance between a given brightness configur-

ation and a given tone configuration is scaled, the amount of transfer from one to the other can be predicted. Generally, two configurations involving only brightness would be closer to each other than would one configuration involving brightness and another involving loudness. Consequently the transfer to a configuration made up of a different class of stimuli would be expected to be fairly small, but it might occur.

It is believed by some writers that a discrimination is learned very quickly when the subject once hits upon the hypothesis which the experimenter intended should be learned. This sudden learning is sometimes called insight. Insight, in terms of the present discussion, would consist of a sudden shift in the distance separating two configurations. An emphasis upon some aspects of the configurations might make two of them seem very close together; an emphasis upon some other aspect might make their apparent distance much greater. A sudden increase in the perceptual distance between two configurations would result in a sudden decrease in difficulty of the problem. This change would produce the insight type of learning curve.

The sudden appearance of transfer is sometimes referred to as an illustration of insight. The ape, for example, may for a time fail to respond to the limb of a tree as a stick for reaching, and then suddenly use it for this purpose. In terms of the present theory this transfer would be described as an increase in the psychological similarity between the limb on the tree and the stick on the floor.

Either of the two foregoing illustrations of "insight" may be regarded as a shift in the "psychological distance" between the two stimuli involved. It is obvious that these statements do not constitute an explanation of insight but merely a suggestion for experimental work. For example, it may be pointed out that a change in the "distance" between two stimulus configurations cannot take place without a simultaneous change in the distances between all similar configurations. Thus, after observing any evidence of insight one would expect shifts in the ease of learning and transfer in numerous related problems. On this point no experimental work has been reported and the possibility seems not to have been noted.

Possible Changes in the Theoretical Formulation

We have chosen certain definitions of the two basic concepts of stimulus-configuration and of response, and have made three assumptions regarding the course of learning. These definitions and postulates were chosen because they seemed to predict more experimental facts and relationships upon a more parsimonious postulational basis than did any other set of definitions and postulates. Others could have been used, and it is possible that some of these might have led to a more general theory, to one with simpler equations, or to one which predicted more experimentally verifiable facts.

Consider, for example, the definitions of stimulus and response used by Spence (25, 26). He has assumed that at the beginning of training the rat has tendencies to respond directionally (to the right or left) and relatively (to the larger or smaller). The relational tendencies appear quite reasonable upon the basis of experimental observation of the behavior of rats. Conversely, it seems unreasonable from an anthropomorphic point of view to deny that the rat discriminates the relative size or brightness of stimulus objects. Yet we have made that denial and defined the rat's response directionally. We rejected Spence's definition and adopted a directional one when we found it possible to predict relational transfer on the basis of directional habits. This prediction seemed scientifically more interesting than to assume a relative tendency at the beginning and then deduce relative transfer later.

The fact that relative transfer can be predicted without assuming that the animal can discriminate relative differences initially shows clearly that the fact of relational transfer can no longer be used as evidence to support the assumption of relational learning. Such prediction suggests a method of ascertaining just what type of evidence is necessary to determine the basis of the rat's discriminatory ability. If human beings can react directly to relative differences in a sense in which rats can not, attempts to verify our deductions regarding transfer should show different results for rats and human beings. These differences might indicate the type of evidence necessary to demonstrate an ability to react directly to relations.

In plotting stimulus configurations we have represented each one by a single point, the two coordinates of which represent the same characteristic (such as size) of the two different stimuli which make up the configuration. This technique is more or less artificial in that it treats a characteristic of each of two stimuli which *happen* to be presented simultaneously as two dimensions. It may be that the theory and the associated scaling technique will apply better to a situation in which a single stimulus object may be described in terms of two characteristics. The discrimination of colors which vary simultaneously in brightness and saturation is one such situation.

Our first two postulates assumed that the learning parameters, c and k, are constant throughout the course of learning; that is, that each trial produces the same learning effect. Krechevsky (16) named this assumption "the continuity theory," and devised an ingenious experiment to show that it did not apply during the initial trials. He suggested that in the first trials the rat is not learning the problem the experimenter had in mind and that consequently rewarding or punishing the rat for correct or incorrect choices has no effect on the

learning of that particular problem. He named this suggestion "the discontinuity theory." The suggestion that the first trials in the discrimination experiment are peculiar and not a part of the same continuum as the remaining ones has previously been made by Lashley (17) and also by Thurstone (29) and by Wiley and Wiley (32) with special reference to the treatment of these errors in rational learning equations.

It is possible to interpret the early trials in a discrimination problem on any one of several bases. The discontinuity theory of Krechevsky is one possibility. Two other assumptions harmonize well with Krechevsky's data. It may be assumed that the early trials constitute a period of adaptation to the experimental situation and that learning does not begin until some time after training has started. Or it may be assumed following Spence (25, 26), that the values of c and k vary with practice.

Finally, it may be assumed, as we have done, that the values of c and k are constant throughout the course of training. The learning curve based on this assumption fits learning data well. The theory based upon it predicts a number of known experimental findings. Without it, the equations are quite complicated. Our assumption is undoubtedly an oversimplification of the real situation. But in the present state of knowledge of learning, the fact that a theory does predict a large number of items concerning learning outweighs in importance the preliminary oversimplification.

Our third postulate was that the slope of the cumulative error curve equaled the ratio of the strengths of the tendencies to respond incorrectly and correctly. While using it gave a number of interesting developments, the equations turned out to be mathematically unwieldy in many places. This assumption was made following Thorndike's suggestion (27) that such a relationship might exist. It has been used because no ready substitute has been found. Possibly the assumption that strength is a negatively accelerated function of the number of rewards and punishments could be combined with a simpler equation relating the slope of the cumulative error curve to the strengths of the responses involved. This combination might lead to a simpler mathematical development, but we have not as yet followed up the possibility.

We realize that the definitions and postulates upon which the theory has been developed will be greeted with skepticism by some psychologists. We are aware that the theory will be criticized because a particular definition or postulate does not agree with the critic's notion regarding the behavior of a learning animal. Criticism on such grounds is not crucial. When an experimental report is published,

other investigators who question the results are at liberty to point out experimental flaws in the method or to repeat the experiment. If the repetition fails to lead to verification of the original results, then the question of which experiment is "right" arises. Either, both, or neither may be; the test is usually not a crucial one. When a rational theory is published, other investigators who question the theory have before them an opportunity for a much more crucial test of its adequacy. The first method is to test the precise deductions drawn from the theory. If these deductions can be shown to be in conflict with experimental data, the theory is shown to be inadequate. The second method consists of developing another theory upon the basis of definitions and postulates which seem more logical to the critic. If the second theory can be shown to be more fruitful — to lead to the prediction of more experimentally known or discoverable relationships, then it has served to render the first theory inadequate. Both methods of attack are open to critics of this theory.

Most of the work reported in the psychological literature consists of single experiments or groups of experiments centered about more or less isolated problems. A given problem is studied; certain results are obtained; and an interpretation is placed upon these results. Someone else repeats the experiment under altered conditions; different results are obtained; and the interpretation is different. Eventually there comes a time when one has a number of groups of experimental results each with a more or less distinct interpretation. This situation is scientifically unsatisfying and so attempts are made to unify the facts with some basic theory. The "identical-elements" theory of transfer of training or Krechevsky's "hypothesis" (15) theory of discrimination learning illustrate the type of intuitive theories which are usually developed when this stage is reached.

The difficulty with intuitive theories lies in the difficulty of devising crucial checks of their adequacy and the limited possibilities of making experimentally verifiable deductions from the theories. Unsatisfied with theories the acceptance of which is so largely a matter of belief and interpretation, some workers have attempted to unify the facts of a given field of experimental knowledge into a rigorous rational theory. Rational theories have a distinct advantage over both isolated experimentation and intuitional theories in that they lead to the deduction of very specific experiments which provide more crucial tests of the adequacy of the theory. The predictive power of such rational theories is illustrated by the developments of Thurstone (28, 29), Hull (11, 12), and of Rashevsky (23). Each of these developments has opened up a number of new experimental problems; each has suggested experimental methods of testing the theory itself.

This theory has presented a number of precise deductions regarding learning and transfer which are capable of quantitative experimental proof or disproof. These experiments will lead to a more comprehensive theory which will render the present one obsolete. In the meantime the present formulation has indicated an order and unity underlying the processes of both learning and transfer.

REFERENCES

The references were printed at the end of the first half of this paper. See *Psychometrika*, Vol. 3, pages 148-149.

ANNOUNCEMENT

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CONTRIBUTION TO THE MATHEMATICAL BIOPHYSICS OF VISUAL PERCEPTION WITH SPECIAL REFERENCE TO THE THEORY OF AESTHETIC VALUES OF GEOMETRICAL PATTERNS

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On the basis of previous mathematico-biophysical studies, a theory of visual perception is outlined. The theory is applied to the aesthetic values of polygonal patterns and found in fair agreement with experimental results.

In a number of previous publications, as well as in our book "Mathematical Biophysics" (1), we have laid down the foundations for a physico-mathematical theory of neuro-physiological and psychological phenomena. This paper is a continuation of the previous studies. A full understanding of the following presupposes a familiarity with the third part of the book, which is referred to as *MB* with a roman figure denoting a chapter.

While the present paper deals in a sense with a theory of the "Gestalt", different from the one discussed in MB, XXVII, it does not reject or contradict the older results. The mechanism of "Gestalt-invariance" discussed in MB and studied further in a paper by A. Householder (2) may account for certain properties, such as perception of wholeness, etc., while other properties may be due to a mechanism discussed here. The two mechanisms, as will be readily seen, may coexist in the central nervous system.

I

In preceding publications (3, 4, 5) we have suggested, how through the mechanism of internal inhibition, a discrimination of absolute intensities of a stimulus may be obtained. We shall now develop the above suggestion somewhat more in detail, indicating a way of putting it on a quantitative mathematical basis.

Let a chain of afferent nerve-fibers carry a peripheral excitation to a center C, in which the end-fiber of the chain gives off a large number of branches, b, all of them forming synapses with neurones (*ne*) of higher order (Fig. 1). Let these neurones possess different

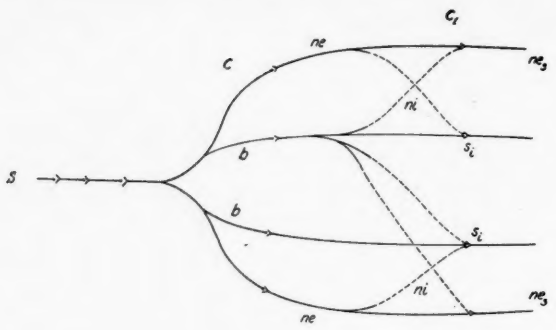

FIGURE 1

thresholds h, distributed according to some distribution-function $N(h)$, so that

$$\int_0^\infty N(h)\,dh = N \tag{1}$$

where N is the total number of neurones (ne).

For a given intensity S of the peripheral stimulus there corresponds a definite value E_1 of intensity of excitation in each branch b (MB, XXII). If the chain consists of only one fiber, then E_1 is a linear function of S. In general however E_1 is a more complicated function of S, depending on the constants of the individual members of the chain

$$E_1 = f(S). \tag{2}$$

The intensity of excitation E_{ne} of each neurone (ne) is given by (MB, XXII):

$$E_{ne} = \beta(PE_1 - h). \tag{3}$$

The total intensity of excitation of all neurones (ne) which have a threshold $h < PE_1$ is then given by

$$E(h) = \beta(PE_1 - h)N(h). \tag{4}$$

If $N(O) = O$ as should be expected from general considerations, then $E(h)$ is zero for $h = O$ and for $h = PE_1$ and is positive within this interval. Hence in that interval $E(h)$ has at least one maximum. Let us first consider the case that it has only one maximum, $h = h_m$, h_m being a function of E_1.

Let all neurones (ne) send fibers to a center C_1, in such a way that all neurones (ne) with the same threshold h form a synapse with

the same neurone of third order (ne_3) and let moreover each neurone (ne) send off a collateral, which excites inhibitory fibers (ni), leading to all the synapses s_i (MB, XXII). Consider for simplicity that all thresholds h_i of ni are the same. In Fig. 1 only one fiber of each group of the same threshold is shown, for simplicity. The situation is formally the same as represented by *equ.* (74) of Chapt. XXII of MB., in which $f_{ki} = Const.$ Replacing the sum by an integral, we have for the value of $(\varepsilon - j)$ at each synapse $s_i(h)$

$$(\varepsilon - j)(h) = PE(h) - Q \int_0^{PE_1} [E(h) - h_i]\, dh . \tag{5}$$

The integral at the right-hand side of (5) is a positive constant, entirely determined by P, E_1 h_i and $E(h)$. Denoting this constant by K, we have

$$(\varepsilon - j)(h) = PE(h) - QK . \tag{6}$$

From what we have said about $E(h)$ and from (6) it follows that $(\varepsilon - j)(h)$ will be positive only for values of h, which lie in the interval (h_1, h_2) where h_1 and h_2 are the smaller and larger roots of the equation

$$E(h) - \frac{QK}{P} = O . \tag{7}$$

h_1 and $h_2 > h_1$ are both positive and lie in the interval (O, PE_1); $h_2 - h_1$ decreases with increasing K. We also have

$$h_1 < h_m < h_2 . \tag{8}$$

If K is large enough, then $(\varepsilon - j)$ at s_i will be positive only for such values of h which lie in the neighborhood of h_m, which correspond to the maximum of $E(h)$. Therefore, although for a given E_1 all neurones (ne) are excited, only a small group of synapses s_i will be excited. If by varying the intensity S of the peripheral stimulus, we vary E_1, this will result in a variation of h_m and if for a new value E'_1 of E_1, the corresponding h'_m will be sufficiently different from the h_m, then entirely different groups of synapses s_i will be excited by the stimulus S' than by S. Thus, to any intensity S of the peripheral stimulus corresponds the excitation of a definite group of synapses s_i. If each of these synapses is connected to a conditioning circuit, described in a previous paper (5), then these synapses may be conditioned to a given reaction. Thus a stimulus of intensity S will produce a reaction R through a group s_i of synapses. A stimulus of a different intensity S' will however not produce R, because it involves a totally different group s'_i of synapses. Each intensity S of the same

stimulus has thus a representative individual group of synapses in the nerve-centers and therefore *each intensity may be* in a way *considered as a different stimulus-pattern.*

However things will happen in this fashion only if the difference between the intensities S and S' is sufficiently large. If this is not the case, then h_m and h'_m differ very little, and the corresponding intervals (h_1, h_2) and (h'_1, h'_2) will partially overlap (Fig 2). If now S is

FIGURE 2

conditioned to a reaction R, then S' produces also R without being previously conditioned to it, through all the synapses, which correspond to the interval (h'_1, h_2) (Fig. 2). Since, because of equa. (4) and (7) h_1 and h_2, as well as h_m are functions of E_1 or, what amounts to the same, functions of S, therefore the minimum difference $\Delta S = S - S'$ for which the intervals (h_1, h_2) and (h'_1, h'_2) do not overlap at all, is itself a function of S. The determination of this function $\Delta S = U(S)$ is in principle a simple problem, which may however involve some complicated algebra. The function $U(S)$ is determined by $f(S)$ in equ. (2) and by $N(h)$ in equ. (1). Several problems suggest themselves at this stage. For instance we may ask the following question: How should $f(S)$ and $N(h)$ be chosen in order that $\Delta S = U(S)$ would have a prescribed form, say that of Fechner's law?

When (h_1, h_2) and (h'_1, h'_2) partially overlap, we may investigate what will be the relative intensity of R, as produced by S' compared to that produced by S. In the simplest case this relative intensity will be given by the ratio η of all neurones lying in the interval (h'_1, h_2) (Fig. 2) to those lying in the interval (h_1, h_2). That is

$$\frac{R'}{R} = \eta = \frac{\int_{h'_1}^{h_2} N(h)\,dh}{\int_{h_1}^{h_2} N(h)\,dh} \tag{9}$$

If we make more complicated assumptions about the possible interaction of the neurones which lie on the efferent side of the synapses s_i we shall obtain expressions different from (9). Here is another interesting group of problems.

The case when $E(h)$ has several maxima in the interval (O, PE_1) is more complicated but is treated in a similar way.

II

We have seen elsewhere (MB, XXVI), that when a fiber, characterized by

$$B_j < A_j\,,\ b_j < a_j\,,\ \frac{A_j}{a_j} < \frac{B_j}{b_j} \tag{10}$$

forms a synapse with another fiber, then the latter becomes excited for a short time only when the intensity of excitation of the first fiber *suddenly increases*. If the first fiber is excited *continuously* with a constant intensity E, the second fiber is not excited. As we have seen in MB the intensity of excitation of the second fiber increases with increasing ΔE where ΔE is the amount of sudden increment of E. As can be readily seen from the discussion in MB, XXVI, the liminal ΔE which still produces an excitation of the second fiber is a linearly decreasing function of E. Since E is in general some function $E = f(S)$ of the intensity of the peripheral stimulus, which involves the first fiber, therefore this liminal ΔE will be in general more or less complicated function of S, depending on the shape of $f(S)$.

If the second fiber leads to a center C_1, discussed in section I, then to each intensity of excitation of the second fiber there will correspond a definite group of excited neurones (ne_3). And since a definite intensity E^* of excitation of the second fiber corresponds to a definite relative sudden increment $\Delta E/E$ of the intensity of excitation of the first fiber, while this relative increment corresponds itself to a definite relative increment $\Delta S/S$ of the peripheral stimulus, therefore to every value of $\Delta S/S$ there will correspond a definite individual group of neurones (ne_3). As has been shown in MB, XXVI, a similar mechanism is obtained for a *sudden* decrease of E, respectively S, that is for a negative $\Delta S/S$.

We thus have a situation, in which a *sudden change* of intensity of a constant external stimulus results in a short excitation of a definite group of central neurones, different groups corresponding to different amounts of change.

Combining this result with that of section I, we may say, that to each absolute value of intensity of a peripheral stimulus corresponds the excitation of a definite center consisting of a definite group of individual neurones ne_3 ; and to each value of relative change of the intensity of peripheral excitation there also corresponds the excitation of a similar definite center. The finite though very large number of neurones in the brain causes an overlapping of some of these centers and thus causes the perception of continuity of the possible values of intensity of peripheral excitation.

III

We shall now introduce some neuro-physiological assumptions, which may appear to be rather questionable. We are introducing them here not because we consider them to be any more likely than other possible assumptions, but rather in a tentative way, as a sort of working hypothesis, in order to show how by means of such or similar assumptions we may develop a physico-mathematical theory of some such phenomena of visual perception, the discussion of which hitherto frequently remained on a purely qualitative level. In the last section of this paper we shall discuss some possible modifications of these hypotheses, modifications which, however, may leave the fundamental formal results unaltered.

When we look at a segment of a straight line, we fix alternately our attention on different points of this line. This results in movement of our eyes along that segment back and forth. Experiments on eye movements (6) show that there is no simple relation between the shape of a figure contemplated and the path of the eye movement. Actual following of a rectangle gives a rather irregular path for the eye movement (7). Inasmuch however as the eye muscles are

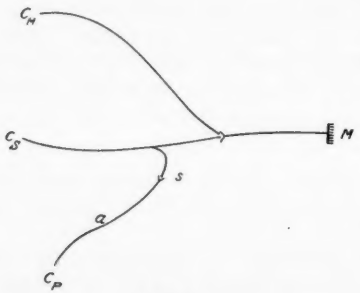

FIGURE 3

innervated by several centers in a rather complicated fashion, the following hypothesis will not necessarily be at variance with observations. Suppose that there is a group of brain centers, which innervate the eye-muscles in such a way as to make the eye follow *exactly* a contour which is projected on the retina. Each eye muscle is however innervated by a number of other centers, which produce different movements, superimposed on the above. Thus the actual movement of the eye may have no apparent relation to the shape of the contour contemplated, yet it contains *as its part* such movements, the path of which is an exact replica of the geometrical pattern looked at.

On Fig. (3) C_s represents schematically the center, which would produce such exact movements of the eye, through the muscle M, while C_M stands for all other centers which may impart to M movements that have nothing to do with the shape of the contour. The character of the movement which would be produced by M under the influence of C_s alone, depends on the character of excitation of C_s (intensity, duration). If through a synapse s a fiber a is excited which leads to a sensory center C_P (Fig. 3), then to each definite type of excitation of C_s, or in other words, to each definite type of movement produced by M under the influence of C_s alone, corresponds a definite intensity and duration of excitation of C_P. If M would be innervated by C_s only, then the same type of excitation in C_P could be obtained through proprioceptive fibers, which would lead from M to C_s, and which would be excited by the contraction of M. In such a case the proprioceptive impulses would have a definite relation to the shape of the contemplated pattern. Actually, due to the other centers C_M, the proprioceptive impulses, coming from M have no relation to the shape of the pattern. But the impulses coming through s over a (Fig. 3) do have such a relation. In other words according to the scheme represented on Fig. (3) things will happen at C_P in such a way, *as if* the movements produced by M would exactly correspond to the shape of the contour contemplated, and *as if* these movements would send off proprioceptive impulses directly to C_P.

The above considerations therefore enable us to speak, for simplicity, of the eye actually following the contemplated contour, and of proprioceptive impulses *corresponding to such movements* being sent to some sensory centers. Though actually things are much more complex, by means of the above scheme we may interpret our schematized statements in terms of a more exact neuro-physiological picture. Thus when in the following we say: "The contraction of the eye muscle sets up proprioceptive impulses, which behave so and so", we do not mean it literally but actually imagine a scheme like the one represented on Fig. 3.

Neurological mechanisms that would produce such a movement can be readily suggested and shall be discussed in a separate paper. For the time being let us make for simplicity and as a special purely theoretical case, the assumption that this movement of the eyes goes on with a constant velocity v. Let a line segment AB be placed in a vertical plane, and let its angle with the vertical be θ. (Fig. 4). Neglecting the possible role of the oblique eye-muscles, the movement of the eyes in the direction AB involves a contraction of the Rectus superior of both eyes, the right Rectus lateralis externus and the left Rectus lateralis internus and a relaxation of both Recti

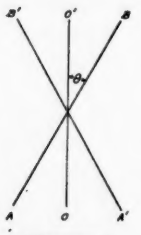

FIGURE 4

inferiors, the right lateralis internus and left lateralis externus (6).

Assuming again for simplicity that the coordinated movements of the left lat. int. and the right lat. ext. are controlled from one common center, and that the same holds for the right lat. int. and left lat. ext., we may greatly simplify our considerations by considering only the movements of *one eye*, for instance of the right one. Then the movement in the direction AB (Fig. 4) involves a contraction of Rectus sup. and of lat. ext. and a relaxation of Rect. inf. and lat. int. which may be written in an abbreviated form, thus:

$$AB : rs + ; le + ; ri - ; li - ; \tag{11}$$

The movement in the direction BA involves:

$$BA : rs - ; le - ; ri + ; li + ; \tag{12}$$

The changes in muscular contraction of the four muscles produce proprioceptive impulses of different intensities and if the proprioceptive fibers lead to centers C_1, discussed in sect. I, then any given intensity of contraction will produce excitation in a definite individual group of neurones (ne_3). Again for simplicity we shall consider that only positive contractions produce proprioceptive impulses, a relaxation being characterized by a lack of such impulses. Let the intensity of these impulses be proportional to the *velocity* of the contraction of the muscles. Then, if v is the velocity of the eye movement, the velocities of contraction of rs and le in (11) are respectively:

$$v \cos\theta \quad \text{and} \quad v \sin\theta$$

and the corresponding intensities of the proprioceptive impulses are:

$$E_{rs} = a\, v \cos\theta \; ; \quad E_{le} = a\, v \sin\theta \tag{13}$$

a being a coefficient of proportionality.

The excitation E_{rs} excites a definite individual group of neurones (ne_3) in a center, which we shall call V (for vertical); while E_{le} excites a group of neurones in a center E (horizontal).

The movement of the eye in the direction BA produces the following excitation:

$$E_{ri} = av \cos\theta \; ; \quad E_{li} = av \sin\theta \tag{14}$$

and causes the excitation of two other groups of neurones (ne_3), one in V, one in E. If the eye moves up and down indefinitely altogether 4 groups of neurones are excited. If we consider now another segment of a straight line, which is characterized by a different $\theta = \theta'$, again 4 groups of neurones (ne_3) will be excited, but these 4 groups will be different from the former 4, because E_{rs}, E_{le}, E_{ri}, E_{li} are different. Some of the new groups may partly overlap with the old ones only when $\theta - \theta' = \Delta\theta$ is very small. Thus to every position of a straight line segment in a vertical plane corresponds the excitation of four distinct individual centers, 2 in H and 2 in V.

Considering now the segment $A'B'$ (Fig. 4), symmetrical to AB with respect to the vertical OO', we shall find by similar considerations, that its scanning with the eye in the direction $A'B'$ gives

$$E'_{rs} = av \cos\theta = E_{rs} \; ; \quad E'_{li} = av \sin\theta = E_{li} \tag{15}$$

and in the direction $B'A'$

$$E'_{ri} = av \cos\theta = E_{ri} \; ; \quad E'_{le} = av \sin\theta = E_{le} \, . \tag{16}$$

That is AB and $A'B'$ *will excite identical groups of neurones* (ne_3) in E and V. This difficulty can be avoided in several ways, of which we shall consider here only one.

Consider a center C_0 of permanent excitation, such as discussed in MB, XXIV, which excites inhibiting fibers, that lead to synapses s, which connect the afferent fibers from li, le and ri to neurones of higher order (Fig. 5). Let the *contraction* of rs excite the fiber (1), which leads to the centers V. Let (1) send off collaterals (2), which

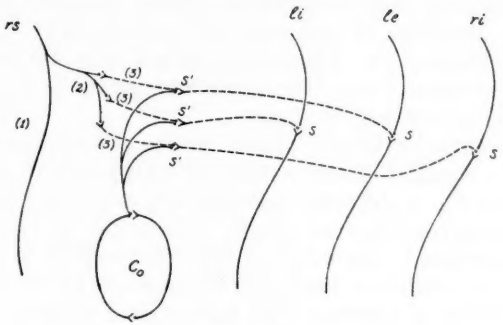

FIGURE 5

excite inhibitory fibers (3), which in their turn inhibit the synapses s' and thus disinhibit s. In that case a contraction of rs, that is a movement of the eye upwards, if followed by a lateral movement produces an excitation both in V and E. But a downward movement of the eye, produced by a contraction of ri and a relaxation of rs, does not result in any excitation in either V or E. Of course other brain centers may be excited by such movement, by means of other fiber connections. But at present we are interested only in V and E. We now have for the movement along AB (Fig. 4).

$$AB : rs+ ; le+ ; ri- ; li- \tag{17}$$

and for BA :

$$BA : rs- ; le- ; ri- : li- \tag{18}$$

For $A'B'$ we have

$$A'B' : rs+ ; li+ ; ri- ; le- \tag{19}$$

and

$$B'A' : rs- ; li- ; ri- ; le- \tag{20}$$

From (17) and (18) it follows that scanning of AB in both directions results in:

$$E_{rs} = av \cos\theta \; ; \quad E_{le} = av \sin\theta \tag{21}$$

while scanning $A'B'$ in both directions results in:

$$E'_{rs} = av \cos\theta \; ; \quad E'_{le} = av \sin\theta \tag{22}$$

Both AB and $A'B'$ involve rs in the same amount and therefore excite the same group of neurones in V. But, while AB involves le, $A'B'$ involves li. Hence the groups in E, due to following of the two lines in both directions, will be different. Every segment of the straight line excites one group of neurones in E and one in V, the groups differing for different θ's. Lines symmetrical with respect to either a vertical or horizontal line have a common group in V. Parallel segments produce identical excitation.

The proprioceptive excitations, due to the contemplation of any segment of a straight line may thus, through V and E, be conditioned to a response R. A different segment will in general not produce R, because it involves different groups of neurones in V and E. A segment symmetrical to the original one with respect to a vertical line may however produce a weaker R through the group of neurones in V, which it has in common with the original segment (MB, XXV).

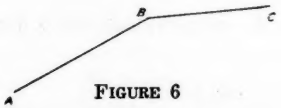

FIGURE 6

Now consider two segments AB and BC (Fig. 6), forming an angle at B. Following this figure with the eye results in the excitation of two groups of neurones in V and of two groups in E. If however the proprioceptive fibers, leading from rs, li, and le to V and E, send off collaterals, which are characterized by (10) and which each leads to a center A^*, A^* will become excited every time the eye passes the angle B, because of a sudden change in the intensity of excitation E_{rs} and E_{le} (or of E_{rs} and E_{li} for a differently oriented angle).

To each value of the angle B corresponds a different intensity of excitation in A^* and if A^* is connected to a center A, of the kind discussed in section I, then to each value of the angle B there will correspond a definite group of neurones (ne_3) in A. We thus have a center for the perception of angles.

Making different hypotheses concerning the physical characteristics and constants of all centers involved, we may derive various quantitative relations, which may suggest experimental studies. As an illustration we shall discuss here only one example. Let A^*_1 be the center, which is excited by a sudden change in $E_{rs} = av \cos\theta$ and let the properties of this center be such that the liminal change ΔE_{rs}, which produces the excitation of A^*_1, is a constant independent on E_{rs}. Since E_{rs} decreases with increasing θ, that constant must be negative. Hence:

$$\Delta E_{rs} = -K. \tag{23}$$

Then the liminal change $\Delta\theta$, which is still perceivable, is given by

$$\Delta\theta = \frac{d\theta}{dE_{rs}}\Delta E_{rs} = -K\frac{d\theta}{dE_{rs}}. \tag{24}$$

Since

$$\theta = \arccos\frac{E_{rs}}{av}, \tag{25}$$

we find:

$$\Delta\theta = \frac{E_{rs}}{av}\frac{K}{\sqrt{1-\frac{E_{rs}}{a^2v^2}}} = \frac{K\,E_{rs}}{\sqrt{a^2v^2 - E^2_{rs}}}.$$

or expressing E_{rs} again through θ:

$$\Delta\theta = K\cot\theta. \tag{26}$$

This equation obviously does not hold for $\theta = 0$, since the $\frac{d\theta}{dE_{rs}} = \infty$,

and (24) cannot be used. Expanding cos θ for very small θ's, we have

$$\cos\theta = 1 - \frac{\theta^2}{2}. \tag{27}$$

Hence

$$E_{rs} = av\ (1 - \frac{\theta^2}{2}) \tag{28}$$

and since for $\theta = 0$, $E_{rs} = av$, $\Delta\theta$ corresponding to $\Delta E_{rs} = K$ is given by:

$$av\ [1 - \frac{(\Delta\theta)^2}{2}] = av - K;$$

$$\Delta\theta = \sqrt{2K}. \tag{29}$$

In the neighborhood of $\theta = 0$, $\Delta\theta$ is given by (29) and is constant. For larger θ's it is given by (26). This simple theory requires that we should easier perceive a small kink in a horizontal line ($\theta = \frac{\pi}{2}$) than in a vertical. Modifications of the assumptions about ΔE_{rs} as well as of other details of the picture, lead to different formulae for $\Delta\theta$. Consideration of the variation of both E_{rs} and E_{li} (or E_{le}) leads to an expression for $\Delta\theta$, which has a minimum for $\theta = 45°$. A number of mathematical problems are thus suggested, which, when worked out and compared with available experimental data, may throw valuable light on our interpretation of the structure of the nerve centers.

Finally we may consider that not only the speeds of the contraction of *rs*, *li* and *le* control the excitation of some centers, but that special fibers are provided, in which the intensity of excitation is proportional to the amplitude of the muscular movements. In this fashion, by similar considerations as above, we shall arrive at a picture, in which a particular group of nerones (ne_3) is excited in a center L, the group being characteristic for the length of the segment of straight line considered.

IV

If a polygonal contour, consisting of n sides and n angles, is presented to the subject then it follows from the foregoing, that as this contour is followed by the eye, in general n groups of neurones will be excited in V, n groups in H, n groups in A and n groups in L. The intensities of excitation of the different groups, even belonging to the same center, such as H, or V, will be in general different. Two groups of neurones (ne_3) corresponding to two different distinct intensities E, will not only be distinct, but will in general contain different numbers of neurones. If the polygon possesses some symmetry

properties, which result for instance in some of the angles being equal, this reduces the number of distinct groups in A. If there are m equal sides, then they all excite the same group of neurones (ne_3) in L, etc. As the contour is followed by the eye, the excitation of each group comes and goes, discontinuously, occurring only when the attention is concentrated on the particular element (segment), angle, length. If out of n elements, m are identical, the group (ne_3) corresponding to those m elements, k, will be excited on the average per unit time m times more often, than those groups (ne_3), which correspond to elements that have no other identical ones. Hence the *average intensity of excitation* of such a common group, corresponding to m identical elements k, will be m times the excitation due to a single element k.

We may now consider a scheme, discussed in MB, XXII, in which an excitatory fiber leads from neurones of each center H, V, A and L to neurones of higher corresponding centers H', V', A' and L'. This excitatory fiber branches off inhibitory fibers, leading to the neurones of all other centers. We may ask for the total excitation corresponding to a given polygonal contour. If this total excitation is transmitted to the pleasure-center (MB, XXX) then its intensity may be considered as a measure of the pleasantness or of the aesthetic value of a given contour. The problem is perfectly definite and can be treated as in MB, provided the intensities of excitation of all the groups (ne_3) in H, V, A and L are given, in other words provided the distribution function $N(h)$ [equa. (1)] for different centers, as well as the functions given by equa. (2), are specified. For different choices of those functions the problem will be of different degree of mathematical difficulty. A particularly simple case is obtained, when we consider a very special case, that the intervals (h_1, h_2) contain always the same number of neurones, and that approximately the intensity of excitation of any such group, falling in any interval (h_1, h_2) is constant. This case is rather unlikely to occur exactly, but we shall consider it here as a possible approximation. In that case two different elements, for instance angles, are characterized by two distinct groups (ne_3), having however the same total intensity of excitation E_0, when the element is perceived. To m identical elements corresponds one group with an *average* intensity mE_0, since the group is stimulated m times more frequently.

We shall apply the above considerations to a group of polygons, for which G. Birkhoff (8) has calculated quantitative aesthetic values on the basis of some general considerations of complexity and order and for which those values have been measured by R. C. Davis by the rank order method (9) (Fig. 7). Let the intensity E_0 of excitation

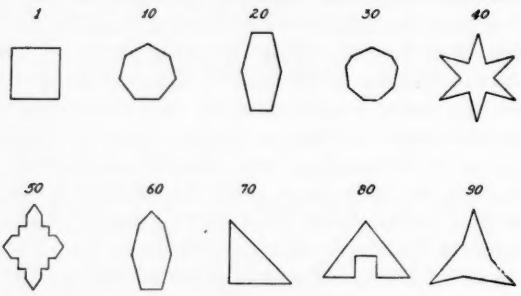

FIGURE 7

of a group (ne_3) be taken as (0.1). The intensity of inhibition, which is approximately proportional to E_0 (MB, XXII) is taken to be 0.001 per unit of E_0. Then for No. 50 (Fig. 8) we have the following:

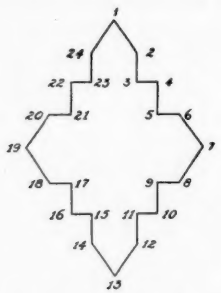

FIGURE 8

There are altogether 8 segments of horizontal lines, which do not give any excitation at V: 3 — 4, 5 — 6, 8 — 9, 10 — 11, 15 — 16, 17 — 18, 20 — 21 and 22 — 23; they all cause the excitation of one group (ne_3) in H, with an intensity of 0.8. The sides 1 — 2, 6 — 7, 13 — 14 and 18 — 19 are all parallel and therefore identical in their H and V groups. Hence in the H center they give one group with $E = 0.4$. Similarly 1 — 24, 19 — 20, 12 — 13 and 7 — 8 give in the H center one group with $E_0 = 0.4$. In the V center the sides 2 — 3, 4 — 5, 9 — 10, 11 — 12, 14 — 15, 16 — 17, 21 — 22 and 23 — 24 give one group with $E_0 = 0.8$; while 1 — 2, 6 — 7, 7 — 8, 12 — 13, 13 — 14, 18 — 19, 19 — 20 and 24 — 1 give another group in V with $E = 0.8$. In the A center we have one group with $E = 1.2$, corresponding to the right angles 3, 4, 5, 9, 10, 11, 15, 16, 17, 21, 22 and 23, a group with $E = 0.2$ for 1 and 13, another group with $E = 0.2$ for 7 and 19, one group with $E = 0.4$ for 2, 12, 14 and 24, and one group with

$E = 0.4$ for 6, 8, 18 and 20. In the L center we have one group with $E = 0.8$ for the 8 equal sides 2 — 3, 4 — 5, 9 — 10, 11 — 12, 14 — 15, 16 — 17, 21 — 22 and 23 — 24, another group with $E = 0.8$ for the 8 equal horizontal sides and a third group with $E = 0.8$ for the 8 equal inclined sides. Altogether we have the following scheme:

H

1 group of $E = 0.8$
3 " " $E = 0.4$

V

2 " " $E = 0.8$

A

1 " " $E = 1.2$
2 " " $E = 0.2$
2 " " $E = 0.4$

L

3 " " $E = 0.8$

The first group of H receives inhibitory fibers from all other groups, that is from altogether 12 groups. The corresponding intensities of inhibition are 0.04, 0.04, 0.008, 0.008, 0.012, 0.002, 0.002, 0.004, 0.004,

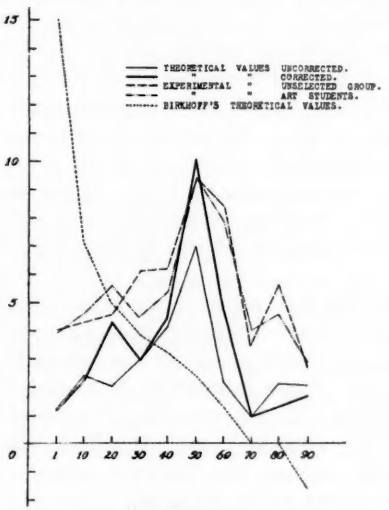

FIGURE 9

0.008, 0.008 and 0.008, altogether 0.072. Hence the net excitation of the first group in V is $0.8 - 0.072 = 0.728$. Each of the two other groups in H receives again inhibitory fibers from 12 other groups, and the net excitation of the group is thus found in a similar way to be 0.324. Calculating thus the net excitations for each group and adding all of them up we obtain for the total value of excitation 7.04. In a similar way the total intensity of excitations of the groups were computed for the other 10 polygons of Fig. 7 and the results are plotted on Fig. 9 (Full light line). The broken and alternate lines of Fig. 9 represent the experimental rank order values of R. C. Davis for two different groups of students. The dotted line represents values calculated by Birkhoff. Inasmuch as our values give the actual intensities of central excitation, measured in arbitrary units, while Davis' values give only the rank order, in which a certain arbitrariness enters in the assignments of weights, we cannot expect a close numerical agreement between our values and those of Davis. The general trend however agrees, with the exception of No. 20.

We notice that our theory gives too low values for No. 20 and No. 60. The fact that No. 60 has actually a much higher value than No. 10, although both are septagons with approximately similar symmetry properties, suggests that the V-groups, which correspond to smaller θ, are more strongly excited than those corresponding to larger θ's. The assumption of unequal excitation of the different groups would make our calculations rather cumbersome. Leaving such a more rigorous study for another paper, we shall approximately take into account a possible preference for vertical directions by multiplying the values of total intensity of excitation as calculated, by the ratio of the heights to widths of the corresponding polygons. This is obviously a crude method, holding only for not too large ratios. The results are shown on Fig. 9, heavy line. The general agreement is somewhat improved, at the expense of No. 80, which is too much lowered. In view of the very crude approximations made, these results should command our attention. Further refinement and generalization of the theory, along lines indicated above, is likely to bring further improvements.

If we do not make the rather crude assumption, that all the centers are excited with equal intensity, then obviously the final results will depend on the assumption about the intensity of excitations of the different centers. These intensities of excitation may vary from individual to individual, thus accounting for the differences in tastes (MB, XXIX). On the average these intensities may be approximately equal, which would account for the fair agreement of the present theory with observations made on larger groups of people.

V

The influence of various symmetries, that is the appearance of several identical elements, may be easily analysed by considerations similar to those given in Chap. XXII of MB. If we have for instance n equally externally excited but different elements, then each of the corresponding groups (ne_3) receives an inhibition from $(n-1)$ other groups. If m of these elements are identical, then we have only $n-m+1$ groups. The total applied excitation remains the same; in the first case it was nE, now it is $(n-m)E+mE$. But the group corresponding to the m identical elements receives inhibitory fibers from $n-m$ other groups. Thus the total inhibition is decreased, but the net excitation increased. If we take as a measure of complexity the total number of elements in the pattern, then various types of symmetry have the effect of reducing the "effective" complexity; but this reduction occurs rather by a process of subtraction than by a process of division, as in Birkhoff's case. It must also be remarked that even when all elements are different, the total excitation in our case does not monotonically decrease with the complexity n. Equation (1) of Chapt. XXIX of MB gives us the intensity of excitatory process at one of the n synapses, corresponding to n different elements with equal intensity of excitation. The total intensity of excitation is obtained by multiplying that equation by n, which gives

$$E_{tot.} = (K_1S - K_4)n - (K_2S + K_3n(n-1)). \tag{30}$$

This expression is zero for $n = 0$ and has a maximum for

$$n^* = \frac{(K_1 - K_2)S - (K_4 + K_3)}{2(K_2S + K_3)}, \tag{31}$$

after which it decreases to zero and becomes negative. We should therefore expect that in a rather simple pattern, consisting of a very few elements, too much symmetry will be rather a disadvantage from the aesthetic point of view, for in that case the total excitation increases with increasing complexity. But for very complex patterns, where the complexity is greater than the optimal n^*, some symmetry is pleasant, as reducing the complexity and bringing it closer to the optimal. In the general case of unequally excited elements, these relations become much more complicated.

If a subject or animal is conditioned to a certain polygonal figure and then another one is presented to him, which has some elements n common with the first one, the intensity of the response to the second figure can be calculated by the method used in Chapt. XXIX of MB. Thus a quantitative theory of discrimination of patterns may be developed. Two patterns will be the more similar, the more common

groups (ne_3) in various centers they have. If all of the neurones in A send off fibers to a center A_0, so that A_0 is always excited, whenever any angle no matter of what size is presented, then A_0 is a center that responds to "angularity" in general. An animal trained to choose a square, when a square and a circle are presented, will choose the more angular figure, when two other but different figures are presented. This provides a basis for the understanding of some interesting results of H. Klüver (10) on the equivalence or nonequivalence of certain pairs of geometrical figures in the behaviour of monkeys. A detailed discussion of this problem must be made the subject of a separate paper.

The mechanism discussed here suggests a definite way for the generalization of the theory, including the aesthetic measure to curvilinear figures. If we again consider a fiber of the kind discussed in section II, but consider instead of a sudden change of E a gradually increasing E, then we shall find by using the fundamental equations developed in MB, XXII that to any given variation of E with respect to time there will correspond a definite intensity of excitation of a definite duration. When a curve is followed by the eye with a definite velocity, then E_{rs}, E_{li} and E_{le} are not constants, but are known functions of time, determined by the shape of the curve. Therefore we may calculate the corresponding intensity of the excitation in a center C_c, to which appropriate fibers are leading. We may then calculate the minimum perceptible curvature and other interesting things. We may also ask, what particular shape of curve gives the maximum excitation at C_c (cf. MB, XXII).

VI

Experiments by Edmund Jacobson (11) indicate that thinking of a vertical object, while not causing actual muscular contractions, yet sets up very weak action currents in the eye muscles, such as would correspond to a vertical movement of the eye. Thinking of horizontal objects gives currents corresponding to horizontal movements. This suggests that also proprioceptive impulses of the kind discussed in section III may be set forth without the muscles performing the actual movements. It has been observed that complete paralysis of the eye muscles does not affect recognition of shapes. This may be due to the taking up of the functions of the eye muscles by other muscles of the body, which may imperceptibly move as a whole, or it may involve a mechanism of gestalt-recognition, discussed previously (MB, XXVII).

Finally, it may be shown, that a mechanism based on our fundamental equations developed in MB can be conceived, in which, when

a chain of neurones is excited along a contour line in a brain center, a wave of excitation is set up, moving along that chain with constant velocity. The vertical and horizontal component of the velocity of that wave may then formally play the same role as the vertical and horizontal movements of the eye. Such mechanisms will be discussed separately. If such a mechanism is at work, then the central wave may continue to travel even for some time after the visual object is removed from the field of vision.

The main thing is that we have here a picture which provides for the analysis of complex patterns, and for the recognition of its elements, and gives also a basis for a quantitative exact theory. Most likely this mechanism is operating simultaneously with the one discussed before (MB, XXVII). The latter provides for recognition of gross features and perception of "wholeness", the former for analysis of details.

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CONDITIONING CIRCUITS

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A circuit of mutually exciting neurones capable of approaching and maintaining a state of permanent excitation has been described by Rashevsky as a possible mechanism for the production of conditioned reflexes. The mode of approach of such a circuit to the steady state is examined in some detail under slightly more general assumptions, and an estimate is made of the order of magnitude of the time required to reach this state.

Rashevsky has proposed a simple neural mechanism for conditioning and has described qualitatively some aspects of its behavior [**3**, chap. XXV; **2**]. The mechanism consists only of two nerve fibers, each exciting the other, having one synapse in common with a synapse of a chain leading from the conditioned stimulus to the effector organs, and reached also by a nerve leading from the unconditioned receptor (Fig. 1). *A priori*, initiation of excitation in one of the nerves of such a pair could lead to one of three possible results. Most obviously, after a few exchanges the excitation might die out.

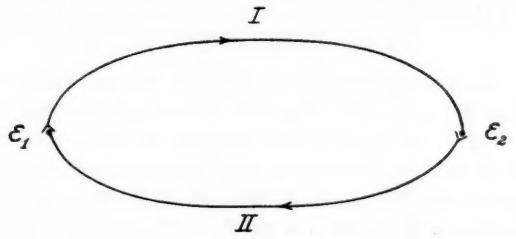

FIGURE I

Or, the excitation might increase with each exchange until it becomes injurious. Or, finally, it might build up, and asymptotically approach some limiting value for each nerve, where a state of permanent excitation is reached. Naturally it is the last case that is of interest, and we wish here to discuss in greater detail the circumstances and the manner in which such an asymptotic state may be approached.

Rashevsky assumes that a stimulus of intensity S applied to a nerve and exceeding a threshold value h, produces a state of excitation in the nerve, this excitation having an intensity E which is a

function of $S—h$. It is assumed that this function $E(S—h)$ has the properties

(a) $E(S—h)$ is properly monotonically increasing;

(b) $E'(S—h)$ is properly monotonically decreasing;

(c) $E(0) = 0$;

(d) $E(\infty) < \infty$.

Property (d) signifies that E has a finite asymptotic value. It is nowhere employed and may be omitted, except for the non-essential equations (7) and the paragraph which follows.

The result of the excitation of the nerve is the production at its end of a substance ε, capable of acting as a stimulus S for any nerve leading from the synapse at which ε is produced. It is assumed that the rate of production of ε is proportional to the intensity E of excitation of the excited nerve, but that ε is dissipated or destroyed at a rate proportional to its own concentration. Thus*

$$d\varepsilon/d\tau = a[E(S-h) - \varepsilon], \tag{1}$$

where the factor a is a property of the substance ε. It is introduced into the function E only for convenience. By a change of time units,

$$t = a\tau, \tag{2}$$

the equation (1) becomes*

$$d\varepsilon/dt = E(S-h) - E. \tag{3}$$

Strictly, when a stimulus S is applied, the production of ε could not start instantly, but only after the nerve impulse has had time to traverse the nerve, so that S and ε must be computed at different times. We shall *assume*, however, that this *propagation time is negligible*. Hence if two fibers are combined to form a conditioning circuit, each ε becomes an S for the other nerve, and we have **[2, 3]**:

$$\begin{aligned} d\varepsilon_1/dt &= E_2(\varepsilon_2 - h_2) - \varepsilon_1, \\ d\varepsilon_2/dt &= E_1(\varepsilon_1 - h_1) - \varepsilon_2. \end{aligned} \tag{4}$$

We have supposed that the *first* nerve is *stimulated by* ε_1 and *produces* ε_2.

Rashevsky proposes for the function E the form

$$E = \frac{I}{\theta}[1 - e^{-a\theta(S-h)}], \tag{5}$$

* For analytical convenience, Rashevsky's proportionality factor A is absorbed into the function E.

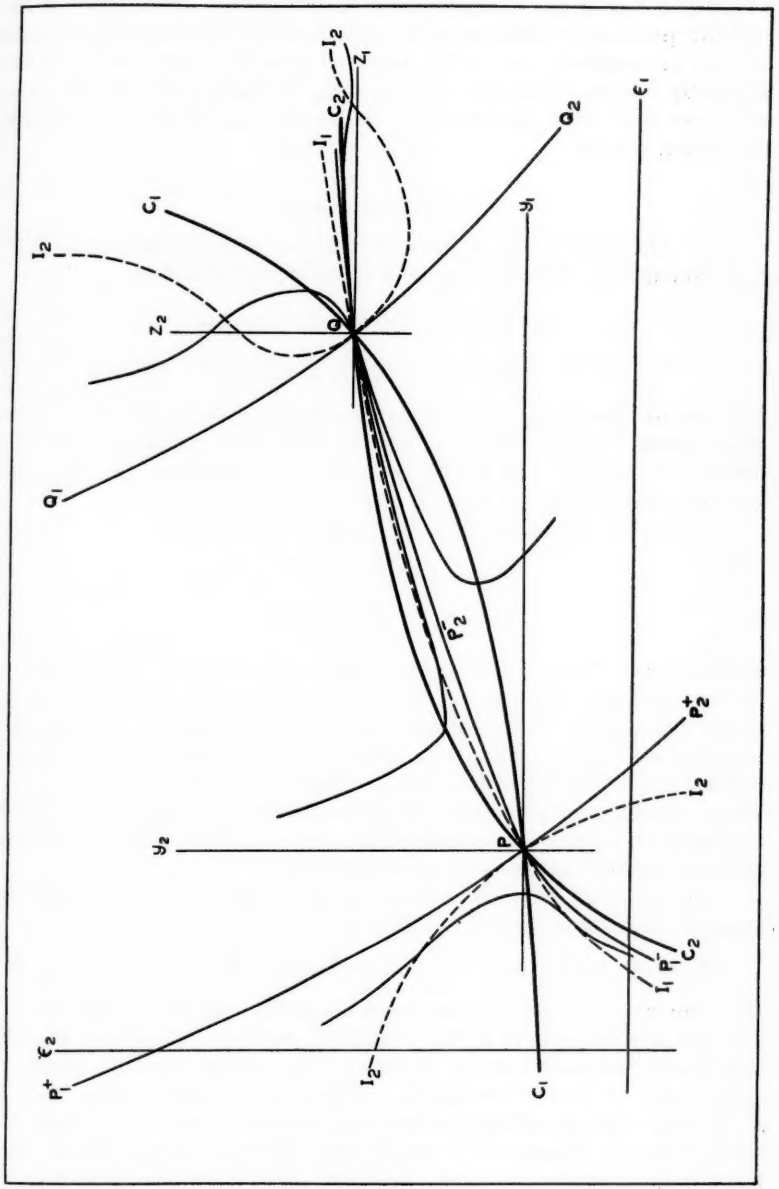

FIGURE 2

where I, θ and α are characteristic of the nerve. However, the important qualitative features of the solutions of (4) are consequences only of assumptions (a) to (d) imposed upon the functions E. Consequently it seems desirable to study first the more general case, after which we shall apply the results to the special case to see what further properties appear.

I. General Case

1. *The Equilibrium Points.* From the form of the functions E, it is evident that the curves C_1 and C_2 (Fig. 2) defined by

$$\begin{aligned} C_1: &\quad \varepsilon_1 = E_2(\varepsilon_2 - h_2), \\ C_2: &\quad \varepsilon_2 = E_1(\varepsilon_1 - h_1), \end{aligned} \tag{6}$$

intersect in two points in the first quadrant, or not at all. We assume these points of intersection, P and Q, to be real and distinct, with P nearest the origin. Let P and Q have the coordinates (p_1, p_2), and (q_1, q_2) respectively.

From (d) we see that C_1 and C_2 have asymptotes

$$\begin{aligned} \varepsilon_1 &= E_2(\infty), \\ \varepsilon_2 &= E_1(\infty), \end{aligned} \tag{7}$$

respectively. Then P and Q lie within the rectangle formed by the two asymptotes and the two axes. Coordinates of these points are easily calculated. Thus, taking the ε_1—axis horizontal, through the intersection of this axis with C_2 draw a vertical line to C_1; from this point draw a horizontal line to C_2; then a vertical to C_1; . . . . . These sequences of abscissas and ordinates approach p_1 and p_2. Starting from an asymptote instead of an axis one obtains q_1 and q_2 to as close an approximation as may be desired.

Evidently at P the slope of C_2 exceeds that of C_1, while at Q the slope of C_1 exceeds that of C_2. Hence

$$E'_1(p_1 - h_1)E'_2(p_2 - h_2) > 1 > E'_1(q_1 - h_1)E'_2(q_2 - h_2). \tag{8}$$

The concave side of the curve C_1 is the region of positive ε'_1, and the concave side of C_2 the region of positive ε'_2. Hence we note that paths (solution curves) drawn in the ε-plane through points in the vicinity of Q tend toward Q, while those in the neighborhood of P, at least through points between the curves C_1 and C_2, are directed away from P. Moreover, P and Q are both equilibrium points since there both derivatives vanish. We may infer, then, that P is a point

of unstable, Q of stable equilibrium. For a more complete analysis we make some transformations.

Let

$$y_1 = \varepsilon_1 - p_1\,, \qquad y_2 = \varepsilon_2 - p_2\,. \tag{9}$$

Then

$$\begin{aligned} y'_1 &= \varphi_2(y_2) - y_1\,, \\ y'_2 &= \varphi_1(y_1) - y_2\,, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \varphi_1(y_1) &\equiv E_1(y_1 + p_1 - h_1) - p_2\,, \\ \varphi_2(y_2) &\equiv E_2(y_2 + p_2 - h_2) - p_1\,. \end{aligned} \tag{11}$$

The functions φ have the properties (a) to (d). Because of (a) we may write

$$\begin{aligned} \varphi_1(y_1) &\equiv \beta^2{}_1\, y_1[1 - \beta_3\, y_1\, \overline{Y}_1(y_1)]\,, \qquad \beta_1 > 0\,, \\ \varphi_2(y_2) &\equiv \beta^2{}_2\, y_2[1 - \beta_4\, y_2\, \overline{Y}_2(y_2)]\,, \qquad \beta_2 > 0\,, \end{aligned} \tag{12}$$

where

$$\overline{Y}_1(0) = \overline{Y}_2(0) = 1\,. \tag{13}$$

Then because of (b)

$$\beta_3 > 0, \qquad \beta_4 > 0\,, \tag{14}$$

while (8) gives

$$\beta_1 \beta_2 > 1\,. \tag{15}$$

From (a) and the fact that $\varphi(0) = 0$, we have

$$1 - \beta_3\, y_1\, \overline{Y}_1 > 0\,, \qquad 1 - \beta_4\, y_2\, \overline{Y}_2 > 0\,. \tag{16}$$

Again from (a) and (b)

$$\begin{aligned} \varphi'_1 &> 0\,, \qquad \varphi'_2 > 0\,, \\ \varphi''_1 &< 0\,, \qquad \varphi''_2 < 0\,. \end{aligned} \tag{17}$$

A similar transformation

$$z_1 = \varepsilon_1 - q_1\,, \qquad z_2 = \varepsilon_2 - q_2 \tag{18}$$

gives

$$\begin{aligned} z'_1 &= \psi_2(z_2) - z_1\,, \\ z'_2 &= \psi_1(z_1) - z_2\,, \end{aligned} \tag{19}$$

with the functions

$$\psi_1(z_1) \equiv E_1(z_1 + q_1 - h_1) - q_2\,,$$
$$\psi_2(z_2) \equiv E_2(z_2 + q_2 - h_2) - q_1\,, \tag{20}$$

having the properties (a) to (d). Again we may set

$$\psi_1(z_1) \equiv \gamma^2{}_1\, z_1[1 - \gamma_3\, z_1\, Z_1(z_1)]\,, \qquad \gamma_1 > 0\,,$$
$$\psi_2(z_2) \equiv \gamma^2{}_2\, z_2[1 - \gamma_4\, z_2\, Z_2(z_2)]\,, \qquad \gamma_2 > 0\,, \tag{21}$$

where

$$Z_1(0) = Z_2(0) = 1\,, \tag{22}$$

and

$$\gamma_3 > 0\,, \qquad \gamma_4 > 0\,. \tag{23}$$

Finally, as above,

$$1 > \gamma_1\,\gamma_2\,, \tag{24}$$

$$1 - \gamma_3\, z_1\, Z_1 > 0\,, \qquad 1 - \gamma_4\, z_2\, Z_2 > 0\,, \tag{25}$$

and

$$\psi'_1 > 0\,, \qquad \psi'_2 > 0\,,$$
$$\psi''_1 < 0\,, \qquad \psi''_2 < 0\,. \tag{26}$$

Now equations (10) can be written

$$y'_1 = -y_1 + \beta^2{}_2\, y_2 - \beta^2{}_2\,\beta_4\, y^2{}_2\, \overline{Y}_2\,,$$
$$y'_2 = \beta^2{}_1\, y_1 - y_2 - \beta^2{}_1\,\beta_3\, y^2{}_1\, \overline{Y}_1\,, \tag{27}$$

and the characteristic equation

$$\begin{vmatrix} -1-\lambda & \beta^2{}_2 \\ \beta^2{}_1 & -1-\lambda \end{vmatrix} \equiv (1+\lambda)^2 - \beta^2{}_1\,\beta^2{}_2 = 0\,, \tag{28}$$

formed from the linear terms, has the roots

$$\lambda_1 \equiv -1 + \beta_1\,\beta_2 > 0\,, \qquad \lambda_2 \equiv -1 - \beta_1\,\beta_2 < 0\,. \tag{29}$$

Then known theorems on differential equations [**1**, pp. 51, 52, 34, 19] show the following (Fig. 2):

(i) *There is a single path* $P_1{}^+$ *from above and a single path* $P_2{}^+$ *from below, each approaching* P *as* t *becomes positively infinite. At* P *these paths both have the tangent*

$$\beta_1\, y_1 + \beta_2\, y_2 = 0\,. \tag{30}$$

(ii) *There is a single path* $P_1{}^-$ *from the left and a single path* $P_2{}^-$ *from the right, each (receding from* P, *i.e.) approaching* P *as* t *becomes* negatively *infinite. At* P *these paths both have the tangent*

$$\beta_1 y_1 - \beta_2 y_2 = 0 . \tag{31}$$

(iii) *Draw a circle with radius arbitrarily small about any point on* P_2^- *as center, and choose any point on* P_1^+ *(or* P_2^+*). Then a circle can be drawn having its center at this point, and a radius so small that every path cutting this circle on the right cuts also the first circle.*

Equations (19) can be written

$$\begin{aligned} z'_1 &= -z_1 + \gamma^2_2 z_2 - \gamma^2_2 \gamma_4 z_2 Z_2 , \\ z'_2 &= \gamma^2_2 z_1 - z_2 - \gamma^2_1 \gamma_3 z_1 Z_1 , \end{aligned} \tag{32}$$

whose linear terms have the characteristic equation

$$\begin{vmatrix} -1-\mu & \gamma^2_2 \\ \gamma^2_1 & -1-\mu \end{vmatrix} \equiv (1+\mu)^2 - \gamma^2_1 \gamma^2_2 = 0, \tag{33}$$

with the roots

$$\mu_1 \equiv -1 + \gamma_1 \gamma_2 < 0 , \qquad \mu_2 \equiv -1 - \gamma_1 \gamma_2 < 0 . \tag{34}$$

Hence [**1**, pp. 50-51]:

(iv) *Through every point within some circle about* Q *as a center, there passes a path approaching* Q *as* t *becomes infinite. It is therefore apparent that through every point above and to the right of the paths* P^+*, the path approaches* Q *as* t *becomes infinite.*

The intersection of one of the paths P^+ with the corresponding ε-axis determines the "threshold of the circuit," h^* [**2**, p. 208]. For let $\bar{\varepsilon}_1$, say, be the intercept of P_2^+ with the ε_1-axis. Then $\bar{\varepsilon}_1$ is the smallest value to which ε_1 can be raised while ε_2 remains zero in order that the circuit shall remain in an excited state. This is immediately evident from (iv). The analytic solution for these paths offers, therefore, a problem of some interest, but it is reserved for a later paper.

Let us next transform to polar coordinates at P by the substitution

$$y_1 = \varrho \cos(\varphi + \omega) , \qquad y_2 = \varrho \sin(\varphi + \omega) , \tag{35}$$

where ω satisfies

$$\sin\omega = \beta_1/\sqrt{\beta^2_1 + \beta^2_2} , \cos\omega = \beta_2/\sqrt{\beta^2_1 + \beta^2_2} . \tag{36}$$

Then

$$\begin{aligned} \varrho' &= y'_1 \cos(\varphi + \omega) + y'_2 \sin(\varphi + \omega) , \\ \varrho\varphi' &= -y'_1 \sin(\varphi + \omega) + y'_2 \cos(\varphi + \omega) ; \end{aligned}$$

$$\cos(\varphi+\omega)=\frac{\beta_2-\beta_1\varphi-\tfrac{1}{2}\beta_2\varphi^2+\cdots}{\sqrt{\beta^2{}_1+\beta^2{}_2}},$$

$$\sin(\varphi+\omega)=\frac{\beta_1+\beta_2\varphi-\tfrac{1}{2}\beta_1\varphi^2+\cdots}{\sqrt{\beta^2{}_1+\beta^2{}_2}}.$$

Hence

$$\begin{aligned}\varrho' &= (\beta_1\beta_2-1)\,\varrho+\cdots,\\ \varphi' &= \beta^2{}_1\beta^2{}_2(\beta_1\beta_4-\beta_2\beta_3)(\beta^2{}_1+\beta^2{}_2)^{-3/2}\varrho-2\beta_1\beta_2\varphi+\cdots.\end{aligned} \tag{37}$$

For these equations the characteristic roots are of opposite signs:

$$\beta_1\beta_2-1>0, \qquad -2\beta_1\beta_2<0.$$

The solution approaching $\varrho=\varphi=0$ as t becomes negatively infinite has, in the limit,

$$d\varphi/d\varrho=\beta^2{}_1\beta^2{}_2(\beta_1\beta_4-\beta_2\beta_3)/[(3\beta_1\beta_2-1)(\beta^2{}_1+\beta^2{}_2)^{3/2}]. \tag{38}$$

Interchanging indices, if necessary, it is no restriction to suppose that

$$\beta_1\beta_4<\beta_2\beta_3. \tag{39}$$

Then the path $P^-{}_2$ departs from P with decreasing φ, and hence with downward concavity.

If we make again the substitutions (35) but with ω defined by

$$\sin\omega=-\beta_1/\sqrt{\beta^2{}_1+\beta^2{}_2}, \qquad \cos\omega=\beta_2/\sqrt{\beta^2{}_1+\beta^2{}_2}, \tag{40}$$

then we have

$$\begin{aligned}\varrho' &= -(\beta_1\beta_2+1)\,\varrho+\cdots,\\ \varphi' &= -\beta^2{}_1\beta^2{}_2(\beta_1\beta_4+\beta_2\beta_3)(\beta^2{}_1+\beta^2{}_2)^{-3/2}\varrho+2\beta_1\beta_2\varphi+\cdots.\end{aligned} \tag{41}$$

In the limit

$$d\varphi/d\varrho=\beta^2{}_1\beta^2{}_2(\beta_1\beta_4+\beta_2\beta_3)/[(3\beta_1\beta_2+1)(\beta^2{}_1+\beta^2{}_2)^{3/2}]. \tag{42}$$

Hence the paths P^+ always have their concavity towards the point Q for large t.

(v) *With assumption* (39) *the paths* P^- *are concave downward at* P, *while in all cases the paths* P^+ *have here their concavity directed toward the point* Q.

Similar transformations may be made at Q, and the results of these can be obtained from the foregoing by merely changing y to z and β to γ. The equations so obtained from (41) give the polar

equations of paths through Q having at Q the tangent

$$\gamma_1 z_1 + \gamma_2 z_2 = 0 . \tag{43}$$

The characteristic equation of the linear term has the roots $-1 - \gamma_1 \gamma_2$ and $+ 2 \gamma_1 \gamma_2$, which are of opposite signs. Hence, there is a unique pair of paths approaching Q with the tangent (43). Then the equation corresponding to (42) shows that at Q these have their concavity away from P.

Equations corresponding to (37) give the polar equations of paths approaching Q having at Q the tangent

$$\gamma_1 z_1 - \gamma_2 z_2 = 0 . \tag{44}$$

Here the characteristic roots are $-1 + \gamma_1 \gamma_2$ and $-2 \gamma_1 \gamma_2$, both negative. Hence there are infinitely many paths approaching Q along this tangent. But every path approaching Q does so along one tangent or the other [**1**, p. 23]. Hence

(vi) *There is a unique path from above and a unique path from below approacing* Q *with the tangent* (43) *as* t *becomes positively infinite. At* Q *these paths are concave away from* P. *Through every point not on one of these paths, but lying above and to the right of the paths* P^+, *there passes a single path which approaches* Q *as* t *becomes positively infinite, which is tangent at* Q *to the line* (44), *and which is concave upward or downward when*

$$(\gamma_1 \gamma_4 - \gamma_2 \gamma_3) / (3 \gamma_1 \gamma_2 - 1)$$

is positive or negative.

The last statement is justified by considering (38) with γ replacing β.

2. *The Inflection Loci.* For more details concerning the form of the paths we consider the points at which their concavity changes, that is, the locus of their inflection points. This is obtained from the determinant

$$I(y_1, y_2) \equiv y'_1 y''_2 - y''_1 y'_2 \equiv z'_1 z''_2 - z''_1 z'_2 \equiv J(z_1, z_2) . \tag{45}$$

Differentiating (10) we have

$$y''_1 = \varphi'_2 y'_2 - y'_1 , \tag{46}$$

$$y''_2 = \varphi'_1 y'_1 - y'_2 .$$

Hence

$$I \equiv \varphi'_1 y'^2_1 - \varphi'_2 y'^2_2 . \tag{47}$$

Since φ'_1 and φ'_2 are always positive, I has the real factors

$$I_1 \equiv \sqrt{\varphi'_1}\, y'_1 - \sqrt{\varphi'_2}\, y'_2 ,$$
$$I_2 \equiv \sqrt{\varphi'_1}\, y'_1 + \sqrt{\varphi'_2}\, y'_2 . \tag{48}$$

Both curves $I_1 = 0$ and $I_2 = 0$ pass through P and Q, but nowhere else intersect either C_1 or C_2.

Near P we have

$$\sqrt{\varphi'_1} = \beta_1 (1 - \beta_3 y_1 + \cdots),$$
$$\sqrt{\varphi'_2} = \beta_2 (1 - \beta_4 y_2 + \cdots), \tag{49}$$

so that

$$I_1 \equiv (\beta_1 \beta_2 + 1)(-\beta_1 y_1 + \beta_2 y_2) + (\beta_1 \beta_2 + 1)(\beta_1 \beta_3 y^2_1 - \beta_2 \beta_4 y^2_2) + \beta_1 \beta_2 (\beta_1 \beta_4 - \beta_2 \beta_3)\, y_1 y_2 + \cdots,$$
$$I_2 \equiv (\beta_1 \beta_2 - 1)(\beta_1 y_1 + \beta_2 y_2) - (\beta_1 \beta_2 - 1)(\beta_1 \beta_3 y^2_1 + \beta_2 \beta_4 y^2_2) - \beta_1 \beta_2 (\beta_1 \beta_4 + \beta_2 \beta_3)\, y_1 y_2 + \cdots. \tag{50}$$

Hence at P the paths P^+ are tangent to I_2 and the paths P^- are tangent to I_1. Likewise at Q, I_2 has the tangent (43) and is therefore tangent to a unique pair of paths, while all other paths approaching Q are tangent at Q to I_1. The curve I_1 passes from P to Q between C_1 and C_2.

Consider a short arc of I_2 containing P, and another short arc containing Q. Call these the P-branch and the Q-branch, respectively. Then these branches of I_2, together with I_1, divide a suitable region of the plane about P and Q into 6 sub-regions. Above the curve I_1 the function I_1 is positive; below it is negative. Between the branches of I_2, the function I_2 is positive; outside it is negative. In the three regions where the signs are alike, I is positive, and the direction of concavity of the path is to the left of the advancing tangent; elsewhere it is to the right.

We shall now show that every path, in crossing I_2, does so in passing from its positive to its negative side. To do this we calculate

$$\partial I / \partial y_1 = \varphi''_1 y'^2_1 - 2\varphi'_1 (y'_1 + \varphi'_2 y'_2),$$
$$\partial I / \partial y_2 = -\varphi''_2 y'^2_2 + 2\varphi'_2 (y'_2 + \varphi'_1 y'_1), \tag{52}$$

giving the components of a vector normal to the curves I_1 and I_2 and extending into the region in which the signs are alike. Along either curve

$$y'_1\, \partial I / \partial y_1 + y'_2\, \partial I / \partial y_2 = \varphi''_1 y'^3_1 - \varphi''_2 y'^3_2 . \tag{53}$$

Now φ''_1 and φ''_2 are always negative, while on I_2 the quantities y'_1 and y'_2 have opposite signs. In particular, on the upper P-branch, $y'_1 > 0$, and hence the expression (53) is negative. Hence at each point of the upper P-branch, the advancing tangent of the trajectory makes an obtuse angle with the normal vector (52). This normal vector here points to the positive side of I_2, and hence the path passes from the positive side to the negative side. With due regard for signs an analogous argument applies at the other arcs of I_2. Otherwise expressed, all paths which intersect I_2 do so in passing, with increasing t, from the region between its two branches into the region outside. It is then geometrically evident that the P-branch, however far extended, nowhere joins the Q-branch. Referring to (v) and (vi) we conclude:

(vii) *The locus of inflection points of the paths is a curve having three branches*: *a branch* $\mathrm{I}_1 = 0$ *containing* P *and* Q *and lying between* C_1 *and* C_2 ; *two branches* $\mathrm{I}_2 = 0$, *one through* P *and one through* Q. *The paths* P^+ *are tangent to the* P*-branch of* I_2 *at* P, *and are elsewhere on the positive side of* I_2. *There is a unique pair of paths tangent, at* Q, *to the* Q*-branch of* I_2, *and lying elsewhere on the positive side of* I_2. *Through every point between these pairs of paths, there passes a path which approaches* Q *from the left tangentially to* I_1 *as* t *becomes infinite. Through every point between the* P*-branch and the pair* P^+ *there passes a path which intersects the* P*-branch and goes off to infinity. Through every point between the other pair and the* Q*-branch there passes a path which intersects the* Q*-branch and then approaches* Q *from the right tangentially to* I_1.

It is of interest to find the direction of concavity of I_1 and I_2 at P, which can be done by setting

$$y_1 = \varrho\,\beta_2\,, \qquad y_2 = \varrho\,\beta_1$$

in I_1, and

$$y_1 = \varrho\,\beta_2\,, \qquad y_2 = -\varrho\,\beta_1$$

in I_2. This gives

$$I_1 = \varrho^2\,\beta_1\,\beta_2(\beta_2\,\beta_3 + \beta_1\,\beta_4) + \cdots.$$

$$I_2 = \varrho^2\,\beta_1\,\beta_2(\beta_2\beta_3 - \beta_1\,\beta_4) + \cdots,$$

From (39) it follows that the tangent to I_1 is on the positive side so that I_1 is concave downward, as is also I_2. At Q, like substitutions in γ show that I_2 is concave upward, but the concavity of I_1 depends upon the sign of $\gamma_2\,\gamma_3 - \gamma_1\,\gamma_4$.

3. *Approach to Steady State.* Consider a pair of equations analogous to (32) of the form

$$z'_2 = -z_1 + \gamma^2_2 z_2 - g_1 ,$$
$$z'_2 = \gamma^2_1 z_1 - z_2 - g_2 , \tag{54}$$

where g_1 and g_2 are functions of t. Let the initial conditions be

$$z_1(0) = \zeta_1 , \qquad z_2(0) = \zeta_2 . \tag{55}$$

It can be verified that these equations have the solutions

$$z_1 = \frac{1}{2\gamma_1}[e^{\mu_1 t}(\gamma_1\zeta_1 + \gamma_2\zeta_2 - J_1) + e^{\mu_2 t}(\gamma_1\zeta_1 - \gamma_2\zeta_2 - J_2)],$$
$$z_2 = \frac{1}{2\gamma_2}[e^{\mu_1 t}(\gamma_1\zeta_1 + \gamma_2\zeta_2 - J_1) - e^{\mu_2 t}(\gamma_1\zeta_1 - \gamma_2\zeta_2 - J_2)], \tag{56}$$

where

$$J_1 = \int_0^t e^{-\mu_1 s}[\gamma_1 g_1(s) + \gamma_2 g_2(s)]\, ds ,$$
$$J_2 = \int_0^t e^{-\mu_2 s}[\gamma_1 g_1(s) - \gamma_2 g_2(s)]\, ds , \tag{57}$$

and μ_1 and μ_2 are given by (34).

If we set

$$g_1 = \gamma^2_2 \gamma_4 z^2_2 Z_2 , \qquad g_2 = \gamma^2_1 \gamma_3 z^2_1 Z_1 , \tag{58}$$

then the g's in (54) are indirectly functions of t through the unknown z's, equations (54) are identified with (32), and (56) is a set of integral equations equivalent to (32) and the initial conditions (55). Explicity, we have

$$\begin{aligned} z_1 = {} & \frac{1}{2\gamma_1}[e^{\mu_1 t}(\gamma_1\zeta_1 + \gamma_2\zeta_2) + e^{\mu_2 t}(\gamma_1\zeta_1 - \gamma_2\zeta_2)] \\ & - \tfrac{1}{2}\gamma_2\{e^{\mu_1 t}\int_0^t e^{-\mu_1 s}(\gamma_2\gamma_4 z^2_2 Z_2 + \gamma_1\gamma_3 z^2_1 Z_1)\, ds \\ & + e^{\mu_2 t}\int_0^t e^{-\mu_2 s}(\gamma_2\gamma_4 z^2_2 Z_2 - \gamma_1\gamma_3 z^2_1 Z_1)\, ds\} , \end{aligned} \tag{59}$$

$$\begin{aligned} z_2 = {} & \frac{1}{2\gamma_2}[e^{\mu_1 t}(\gamma_1\zeta_1 + \gamma_2\zeta_2) - e^{\mu_2 t}(\gamma_1\zeta_1 - \gamma_2\zeta_2)] \\ & - \tfrac{1}{2}\gamma_1\{e^{\mu_1 t}\int_0^t e^{-\mu_1 s}(\gamma_2\gamma_4 z^2_2 Z_2 + \gamma_1\gamma_3 z^2_1 Z_1)\, ds \end{aligned}$$

$$- e^{\mu_2 t} \int_0^t e^{-\mu_2 s} (\gamma_2 \gamma_4 z^2{}_2 Z_2 - \gamma_1 \gamma_3 z^2{}_1 Z_1)\, d s \} .$$

This pair of integral equations is non-linear, but nevertheless for at least those paths which converge to Q an iterative procedure for effecting a numerical solution is possible just as in the linear case. Take the terms outside the integral signs as first approximations $z^{(0)}(t)$ to $z(t)$:

$$z^{(0)}{}_1(t) = \frac{1}{2\gamma_1} [e^{\mu_1 t}(\gamma_1 \zeta_1 + \gamma_2 \zeta_2) + e^{\mu_2 t}(\gamma_1 \zeta_1 - \gamma_2 \zeta_2)] ,$$

$$z^{(0)}{}_2(t) = \frac{1}{2\gamma_2} [e^{\mu_1 t}(\gamma_1 \zeta_1 + \gamma_2 \zeta_2) - e^{\mu_2 t}(\gamma_1 \zeta_1 - \gamma_2 \zeta_2)] . \tag{60}$$

If these values are used to replace the z_1 and z_2 under the integral sign in (59), we obtain second approximations $z^{(1)}{}_1(t)$, $z^{(1)}{}_2(t)$. The quadratures can be made exactly or by any suitable scheme for numerical quadrature. Substituting again with these results we get $z^{(2)}{}_1(t)$, $z^{(2)}{}_2(t)$, and so on sequentially. For arcs near to P it may be preferable to use, instead of (59), the analogous equations in y .

It is desirable in particular to estimate the rate at which Q is approached, or the time necessary to come within an assigned distance from Q . To make such an estimate we first obtain from (56)

$$\gamma_1 z_1 + \gamma_2 z_2 = e^{\mu_1 t}(\gamma_1 \zeta_1 + \gamma_2 \zeta_2 - J_1(t)) . \tag{61}$$

The left member of this equation is $(\gamma^2{}_1 + \gamma^2{}_2)^{1/2}$ times the distance from the line (43) to the point (z_1 , z_2) on the path, and the first two terms in parentheses give the same multiple of the distance from this line to the initial point. For initial points not too far removed from the exceptional path P_2^- running from P for $t = -\infty$ to Q for $t = +\infty$, it is evident geometrically that the distance from this line may be used as a fairly satisfactory measure of the distance from Q . As t becomes infinite the left member of (61) vanishes, and hence so does the right member.

Set

$$\gamma_1 z_1 + \gamma_2 z_2 = z(\gamma_1 \zeta_1 + \gamma_2 \zeta_2) . \tag{62}$$

Then from (61)

$$t = \frac{1}{\mu_1} \log z - \frac{1}{\mu_1} \log \left(1 - \frac{J_1(t)}{\gamma_1 \zeta_1 + \gamma_2 \zeta_2}\right) . \tag{63}$$

Neglecting J_1 amounts to finding the time on the assumption that the approximation $z^{(0)}{}_1$, $z^{(0)}{}_2$ is exact. However, a better value for t can be found by evaluating J_1 (to a sufficiently large t) with $z^{(0)}{}_1$ and $z^{(0)}{}_2$ replacing z_1 and z_2 .

For initial points relatively close to the line (43) a like discussion can be made using $\gamma_1 z_1 - \gamma_2 z_2$.

II. The Case of Exponential Development

For Rashevsky's case of exponential development of excitation a preliminary transformation yields a special symmetry of the equations. We start with the equations

$$\varepsilon'_1 = \frac{A I_2}{\theta_2}[1 - e^{-a_2\theta_2(\varepsilon_2 - h_2)}] - \varepsilon_1 ,$$
$$\varepsilon'_2 = \frac{A I_1}{\theta_1}[1 - e^{-a_1\theta_1(\varepsilon_1 - h_1)}] - \varepsilon_2 , \tag{64}$$

after changing the time unit as in (2) to eliminate the factor a. Then the substitutions

$$x_1 = a_1 \theta_1 \varepsilon_1 , \qquad x_2 = a_2 \theta_2 \varepsilon_2 ,$$
$$c_1 = A I_1 a_2 \theta_2/\theta_1 , \qquad c_2 = A I_2 a_1 \theta_1/\theta_2 , \tag{65}$$
$$b_1 = c_1 e^{a_1\theta_1 h_1} , \qquad b^2 = c_2 e^{a_2\theta_2 h_2} ,$$

reduce the system to the form

$$x'_1 = c_2 - b_2 e^{-x_2} - x_1 ,$$
$$x'_2 = c_1 - b_1 e^{-x_1} - x_2 . \tag{66}$$

The transformation analogous to (9) gives

$$y'_1 = \beta^2{}_2(1 - e^{-y_2}) - y_1 ,$$
$$y'_2 = \beta^2{}_1(1 - e^{-y_1}) - y_2 , \tag{67}$$

where

$$\beta^2{}_1 = b_1 e^{-p_1} = c_1 - p_2 , \qquad \beta^2{}_2 = b_2 e^{-p_2} = c_2 - p_1 . \tag{68}$$

The transformation analogous to (18) gives

$$z'_1 = \gamma^2{}_2(1 - e^{-z_2}) - z_1 ,$$
$$z'_2 = \gamma^2{}_1(1 - e^{-z_1}) - z_2 , \tag{69}$$

where

$$\gamma^2{}_1 = b_1 e^{-q_1} = c_1 - q_2 , \quad \gamma^2{}_2 = b_2 e^{-q_2} = c_2 - q_1 . \tag{70}$$

Evidently

$$\beta_3 = \beta_4 = \gamma_3 = \gamma_4 = 1/2 . \tag{71}$$

Hence convention (39) becomes

$$\beta_1 < \beta_2 . \tag{72}$$

It then follows that

$$\gamma_1 < \gamma_2 . \tag{73}$$

To show this, let

$$r_1 = q_1 - p_1 , \quad r_2 = q_2 - p_2 .$$

Then

$$\gamma^2{}_1 = \beta^2{}_1 e^{-r_1} = \beta^2{}_1 - r_2 , \quad \gamma^2{}_2 = \beta^2{}_2 e^{-r_2} = \beta^2{}_2 - r_1 .$$

If $r_2 > r_1$, then

$$\gamma^2{}_1 - \gamma^2{}_2 = (\beta^2{}_1 - \beta^2{}_2) + (r_1 - r_2) < 0$$

and (73) is true. If, on the other hand, $r_2 < r_1$, then

$$\gamma^2{}_1/\gamma^2{}_2 = e^{r_2 - r_1} \beta^2{}_1/\beta^2{}_2 < 1$$

and (73) still holds. It follows, therefore, that I_1 *is concave downward both at* P *and at* Q.

It can be shown further, in this case, that *paths cross* I_1 *only from above, never from below.* First we have from (48)

$$I_1 \equiv \beta_1 e^{-y_1/2} y'_1 - \beta_2 e^{-y_2/2} y'_2 .$$

Then from (53)

$$\begin{aligned} y'_1 I_{y_1} + y'_2 I_{y_2} &= -\beta^2{}_1 e^{-y_1} y'^3_1 + \beta^2{}_2 e^{-y_2} y'^3_2 \\ &= \beta_1 e^{-y_1/2} y'^2_1 y'_2 (\beta_1 e^{-y_1/2} - \beta_2 e^{-y_2/2}) \end{aligned}$$

along I_1. This quantity is negative and the statement is true at least near P and on the right because of (72) and the fact that all factors outside the parentheses are positive. The quantity in parentheses can change signs on I_1 only at a point at which

$$\beta_1 e^{-y_1/2} - \beta_2 e^{-y_2/2} = 0$$

$$\beta_1 e^{-y_1/2} y'_1 - \beta_2 e^{-y_2/2} y'_2 = 0$$

simultaneously. The curves defined by these equations intersect only on the curve

$$y'_1 - y'_2 = 0 ,$$

or

$$\beta^2{}_2 - \beta^2{}_1 - \beta^2{}_2 e^{-y_2} + \beta^2{}_1 e^{-y_1} - y_1 + y_2 = 0 ,$$

or

$$\beta^2{}_2 - \beta^2{}_1 = y_1 - y_2 ,$$

and hence at a point for which

$$y_1 > y_2 .$$

On the other hand, at such a point

$$e^{(y_1-y_2)/2} = \beta_1/\beta_2 < 1\,,$$

$$y_1 < y_2\,.$$

Thus no such point exists and the sign changes only at P or Q.

Since all paths have the same direction of concavity at Q_2 and some approach Q from below I_1, it follows that all paths [except those two which have the tangent (43)] approach from below. This completes the proof of the above statement. Hence every point above I_1 and to the right of the paths P^+ lies either on the path approaching Q from above along (43), or else it lies on a path which crosses I_1 once and then approaches Q tangentially to I_1.

The functions g of (54) and (57) are

$$\begin{aligned} g_1 &= \gamma^2{}_2(e^{-z_2} - z_2 - 1) = \tfrac{1}{2}\,\gamma^2{}_2\,z^2{}_2 - \cdots\,, \\ g_2 &= \gamma^2{}_1(e^{-z_1} - z_1 - 1) = \tfrac{1}{2}\,\gamma^2{}_1\,z^2{}_1 - \cdots\,. \end{aligned} \tag{74}$$

For negative z_1 and z_2 both are everywhere positive, and hence J_1 is positive. Hence neglecting J_1 in (63) gives always too small a value for t, when ζ_1 and ζ_2 are both negative. This is apt to be the case of greatest interest since the initiation of excitation will normally take place by raising one of the ε's above the threshold of the circuit while the other ε is initially zero. Thus one of the ζ's is certainly negative. The other ζ, if positive, is necessarily small and the corresponding z becomes and remains negative after a short interval t.

Some of the results obtained above are illustrated in Fig. 2. The crossing of I_1 from above, illustrated in the figure, has been proved only for the exponential case. The peculiar shape of the Q-branch of I_2 has not been discussed, but it is qualitatively correct, at least for the exponential case. The general concavity of I is not decided. In all other respects, the essential features are as indicated even in the general case.

This figure is to be compared with Rashevsky's Fig. 2 of **2**, and Fig. 61 of **3**, ch. XXIV. These figures, drawn from general considerations and without detailed analysis, are validated in the main. We note only the following points: Rashevsky does not give the I-curves, and the behavior near the exceptional paths to Q is left undecided; also the curve $A\,B$ is our pair of paths P^+, and should be concave in the opposite direction.

III. The Time Lag

If the propagation time for the two neurones is not negligible, the form of the problem is altered considerably. We shall merely formulate the problem without attempting to develop the properties of

the solutions. Let t be the time that elapses between the application of the stimulus to either neurone and the consequent development of ε. We suppose, for simplicity, that this is the same for both neurones of the circuit. We suppose that the time units have been chosen so that the parameter $a = 1$. Suppose, for example, that initially $\varepsilon_1 = 0$, and that during the time-interval from $t = 0$ to $t = 2a$, ε_2 is developed by an afferent neurone and has at any time t of this interval the value $\varepsilon_2(t)$. Then from the time $t = 0$ to the time $t = a$, there is still no production of ε_1, because of the propagation lag. But from $t = a$ on the development of ε_1 proceeds and at any time t the development is determined by the value of ε_2 at the time $t - a$. Hence, in place of the first of equations (4) we have

$$\varepsilon'_1(t) = A\,E_2[\varepsilon_2(t-a) - h_2] - \varepsilon_1(t)\,. \tag{75}$$

For simplicity let us set

$$\begin{aligned} \eta_2(t) &= A\,E_2[\varepsilon_2(t-a) - h_2]\,, \\ \eta_1(t) &= A\,E_1[\varepsilon_1(t-a) - h_1]\,. \end{aligned} \tag{76}$$

Then, as we have seen, $\eta_2(t)$ is known on the interval from $t = a$ to $t = 3a$. Hence from $t = 0$ to $t = a$, $\varepsilon_1(t) = 0$, while from $t = a$ to $t = 3a$, $\varepsilon_1(t)$ is given as the solution of

$$\begin{aligned} \varepsilon'_1(t) &= \eta_2(t) - \varepsilon_1(t), \\ \varepsilon_1(a) &= 0\,. \end{aligned} \tag{77}$$

This equation is easily solved:

$$\varepsilon_1 = e^{-t}\int_a^t e^s\,\eta_2(s)\,ds\,, \quad a \leqslant t \leqslant 3a\,. \tag{78}$$

But then $\eta_1(t)$ becomes known on the interval from $t = 2a$ to $t = 4a$, and if the afferent neurone is no longer excited (if it is we simply add in the ε_2 which it produces) we find in like manner

$$\varepsilon_2 = e^{-t}\{e^{2a}\,\varepsilon_2(2a) + \int_{2a}^t e^s\,\eta_1(s)\,ds\}\,, \; 2a \leqslant t \leqslant 4a\,. \tag{79}$$

Proceeding sequentially we obtain a set of recursion formulas giving ε_1 on the intervals $(2n-1)a \leqslant t \leqslant (2n+1)a$, and ε_2 on the intervals $2na \leqslant t \leqslant (2n+2)a$.

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THE RELATION BETWEEN THE INTENSITY OF EXCITATION AND THE NUMBER OF NEURONES TRAVERSED

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In previous formulations the assumption has been made that the intensity of excitation or inhibition varies with distance. In this paper the simple case of a series of identical neurones is treated. The development shows five possible relations. In future theoretical formulations it is no longer necessary to postulate arbitrary relations in case any of the derived forms can be used.

In chapter xxii of his book "Mathematical Biophysics" N. Rashevsky (1) discusses how the intensity of nerve excitation, traveling

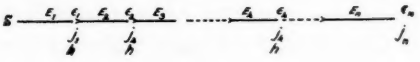

FIGURE 1

along a chain of neurones, varies in a definite manner along the chain, the variation depending upon the constants of the neurones involved. The following development is confined to the case of a series of identical neurones over which a nerve impulse may pass. (Fig. 1). We consider neurones of the general excitatory type (1), producing both ε and j. The excitatory and inhibitory factors ε and j are postulated as being produced at a synapse and their difference, if greater than the synaptic threshold, serves as the stimulus for the higher order nerve. If the chain is an inhibitory one, the last neurone of the series is inhibitory (1).

Let a constant stimulus S be applied to the afferent end of the first neurone of the series. The intensity of excitation in the first neurone is then given by

$$E_1 = a I_0 (S - h), \qquad S > h, \tag{1}$$

where S is the strength of the stimulus, h the threshold, I_0 the intensity of each individual nerve impulse, and a a constant. By hypothesis we have a, I_0, and h the same for each nerve fiber.

Now for the k^{th} neurone the differential equations for the excitation and inhibition factors are

$$\frac{d\varepsilon_k}{dt} = A_e E_k - a_e \varepsilon_k , \tag{2}$$

$$\frac{dj_k}{dt} = B_e E_k - b_e j_k , \tag{3}$$

where we also have

$$E_k = a\, I_0 (\varepsilon_{k-1} - j_{k-1} - h). \tag{4}$$

The solutions of equations (2) and (3) for a constant E_k are

$$\varepsilon_k = \frac{A_e E_k}{a_e} (1 - e^{-a_e t}), \tag{5}$$

and

$$j_k = \frac{B_e E_k}{b_e} (1 - e^{-b_e t}), \tag{6}$$

where we have $\varepsilon_k = j_k = 0$ for $t = 0$. For convenience the origin of the time axis is moved forward with each k. It is necessary to take this into account if the time delays are of interest.

Let us now consider a case in which the stimulus S has been applied sufficiently long so that we may neglect the exponential terms. Then we have

$$\varepsilon_k - j_k = (\frac{A_e}{a_e} - \frac{B_e}{b_e})\, E_k . \tag{7}$$

Define P_1 by the relation*

$$P_1 = (\frac{A_e}{a_e} - \frac{B_e}{b_e})\, a\, I_0 . \tag{8}$$

Then using equations (1), (4) and (7) we have the following relations

$$\varepsilon_1 - j_1 = (\frac{A_e}{a_e} - \frac{B_e}{b_e})\, a\, I_0 (S - h) = P_1 S - P_1 h ,$$

$$\varepsilon_2 - j_2 = (\frac{A_e}{a_e} - \frac{B_e}{b_e})\, a\, I_0 \,[\, (\frac{A_e}{a_e} - \frac{B_e}{b_e})\, a\, I_0 (S - h) - h]$$

$$= P_1^2 S - P_1^2 h - P_1 h ,$$

$$\varepsilon_3 - j_3 = (\frac{A_e}{a_e} - \frac{B_e}{b_e})\, a\, I_0 [\, (\frac{A_e}{a_e} - \frac{B_e}{b_e})^2\, a^2\, I_0^2 (S - h)$$

$$- (\frac{A_e}{a_e} - \frac{B_e}{b_e})\, a\, I_0\, h - h]$$

* P_1 should not be confused with the symbol P used by Rashevsky (1). P was defined by the expression in parentheses in equation (8), or $P = P_1 / a I_0$.

$$\varepsilon_3 - j_3 = P_1^3 S - P_1^3 h - P_1^2 h - P_1 h ,$$

$$\vdots \qquad\qquad \vdots$$

$$\varepsilon_n - j_n = P_1^n S - P_1 (P_1^{n-1} + P_1^{n-2} + \cdots + 1) h , \tag{9}$$

and replacing the geometric series by its sum, we have

$$\varepsilon_n - j_n = P_1^n (S + \frac{P_1 h}{1 - P_1}) - \frac{P_1 h}{1 - P_1}, \quad \text{for } P_1 \neq 1 \tag{10}$$

or

$$\varepsilon_n - j_n = S - n h , \quad \text{for } P_1 = 1 . \tag{11}$$

In Figure 2 the curves for the various cases are shown. For

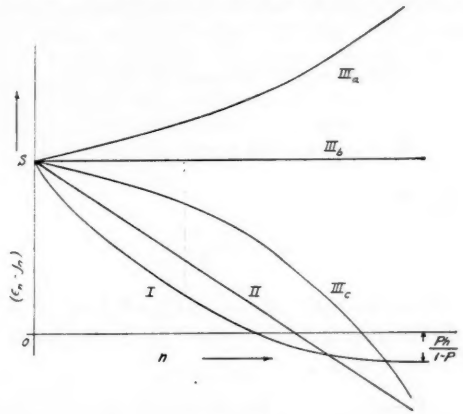

FIGURE 2

$P_1 < 1$ the function decreases exponentially from the initial value S and approaches $-\frac{P_1 h}{1 - P_1}$ (Curve I). For $P_1 = 1$ we have a straight line with a negative slope, $-h$ (Curve II). For $P_1 > 1$ we have three possibilities. If $P_1 > S/(S-h)$, $\varepsilon_n - j_n$ is an exponentially increasing function of n (Curve $\mathrm{III_a}$). If $P_1 = S/(S-h)$, then $\varepsilon_n - j_n = S$, a constant (Curve $\mathrm{III_b}$). If $P_1 < S/(S-h)$, $\varepsilon_n - j_n$ decreases monotonically with n and becomes negatively infinite (Curve $\mathrm{III_c}$).

It may be that the number of neurones between two points is proportional to the distance. Equation (10) or (11) then gives the relation between the net excitation at the efferent end of a nerve fiber as

it varies with the distance of the nerve fiber from an arbitrary origin. The cases I and III_a may be written

$$S(x) = C^2 e^{z|mx|} + d, \tag{12}$$

or if we may neglect d,

$$S(x) = S_0 e^{z|mx|}. \tag{13}$$

An equation of the above form was assumed by Householder and Amelotti (2). Here we see that it is an approximation to equation (12) or two cases of equation (10) and that it is obtained directly from postulating a series of similar neurones. The case III_b is also a special case of equation (13) where $m = 0$. This simplest case was assumed by Rashevsky in treating the theory of delayed reflexes (3).

Let us now solve the equation

$$\varepsilon_{n'} - j_{n'} = h \tag{14}$$

where n' is the value of n at which the excitation wave stops. Substituting equations (10) and (11) into (14) we have, for $P_1 \neq 1$

$$P_1^{n'} S + P_1^{n'+1} h - P_1^{n'+1} S = h, \tag{15}$$

or

$$n' = \frac{1}{\log P_1} \cdot \log \frac{h}{S(1-P_1) + P_1 h}; \tag{16}$$

and similarly for $P_1 = 1$

$$n' = \frac{S-h}{h}. \tag{17}$$

In equation (16), n' is real unless $[S(1-P_1) + P_1 h] \leqslant 0$ (Curves III_a, III_b). Differentiating with respect to S, we have from (16) and (17), for real values of n',

$$\frac{dn'}{dS} = \frac{P_1 - 1}{\log P_1} \frac{1}{[S(1-P_1) + P_1 h]} > 0, \quad \text{for } P_1 \neq 1 \tag{18}$$

since

$$[S(1 - P_1) + P_1 h] > 0,$$

and

$$\frac{dn'}{dS} = \frac{1}{h} > 0, \quad \text{for} \quad P_1 = 1 \tag{19}$$

so that regardless of the values of P_1, n' increases with S.

Let S_1 and S_2 be the magnitudes of two stimuli incident upon the afferent ends of two similar chains of identical neurones. Then it follows from equations (10) and (11) that the difference in stimulus value at the end of the chain is given by

$$S'_1 - S'_2 = \varepsilon_{1n} - j_{1n} - (\varepsilon_{2n} - j_{2n}) = \gamma(S_1 - S_2) \tag{20}$$

where γ is a constant. Since generally we know only the relative values of the stimuli, equation (20) permits the use of $S_1 - S_2$ instead of $S'_1 - S'_2$; that is, we may use the intensity difference at any point along the two parallel series.

In the case of an excitation wave passing over a series of identical neurones, the excitation may increase or decrease with distance, or remain constant, depending on the choice of parameters. Evidently the assumption that any or all of the parameters vary with distance will lead to still more complex relations than those derived above. However, such complications need not be made at present since the simplest case already gives sufficient possibilities.

This investigation has been made possible through a grant from the Rockefeller Foundation to the University of Chicago.

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PSYCHOMETRIKA—VOL. 3, NO. 4
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A NOTE ON DWYER: THE DETERMINATION OF THE FACTOR LOADINGS OF A GIVEN TEST

If a factorial analysis of a battery of tests has been performed, and the centroid or principal axis factor loadings and the primary trait loadings* for each test have been determined, it is sometimes desired to discover the loadings of additional tests on the same primary traits without the necessity of factoring a new correlation matrix consisting of all the inter-correlations of the original battery of tests plus the additional tests.

In a recent article, Dwyer† has presented a method for determining the factor loadings of additional tests. The method as presented by Dwyer is primarily adapted for use with the results of a centroid or principal axis analysis. This note is merely a restatement of Dwyer's method in more condensed formulation, and an extension of it to the determination of primary trait loadings.

a. Let there be:
   n tests, with general subscript j, of known loadings on r factors with general subscript m; let the factor loading of the j^{th} test on the m^{th} factor be a_{mj}; and let $F_{mj} \equiv |a_{mj}|$.

b. Let there be s additional tests, with general subscript t, whose loadings a_{mt} are sought, given the correlation coefficient r_{jt} between each of the s tests, t, and each of the n tests, j; let F_{mt} be the matrix of factor loadings a_{mt}; and let R_{jt} be the matrix of correlation coefficients, r_{jt}.

Referring to page 174 of Dwyer's presentation, and rewriting his second set of equations in the above notation, we have:

$$F'_{jm} F_{mt} = R_{jt} . \tag{1}$$

If both sides of equation (1) be multiplied on the left by F_{mj}, we have

$$F_{mj} F'_{jm} F_{mt} = F_{mj} R_{jt} . \tag{2}$$

* Thurstone, L. L. *Vectors of Mind.* Chicago: University of Chicago Press, 1935, Ch. VI.

† Dwyer, Paul F. "The determination of the factor loadings of a given test from the known factor loadings of other tests," *Psychometrika*, **2**, 1937, 173-179.

Now, under the conditions given in the definition of F_{mj}, the matrix product $F_{mj}F'_{jm}$ is non-singular and has an inverse, whence:

$$F_{mt} = (F_{mj}\,F'_{jm})^{-1}\,F_{mj}\,R_{jt}\,. \tag{3}$$

If we define

$$F_{mj}\,F'_{jm} \equiv G_{mM}\,, \tag{4}$$

then (3) becomes

$$F_{mt} = G^{-1}_{mM}\,F_{mj}\,R_{jt}\,. \tag{5}$$

Now if, as is the case with centroid or principal axis factor loadings, all non-diagonal elements of G_{mM} are sensibly zero, i.e.,

$$g_{mM} = 0\,, \quad (m \neq M), \tag{6}$$

then G_{mM} is a diagonal matrix, and we may write:

$$G_{mM} = D_{mm}\,, \text{where } d_{mM} = 0\,,\ d_{mm} = \sum_{j=1}^{n} a^2_{mj} \tag{7}$$

whence

$$G^{-1}_{mM} = D^{-1}_{mm} = \left| \frac{1}{\sum_{j=1}^{n} a^2_{mj}} \right| \tag{8}$$

If what are ultimately desired are not the loadings on an arbitrary set of centroid axes, but the primary trait loadings, v_{tp}, where

$$V_{tp} \equiv |v_{tp}| = F'_{tm}\,T_{mp}\,, \tag{9}$$

(T_{mp} being a transformation matrix, determined in the previous analysis of the n tests) then

$$V_{tp} = R'_{tj}\,F'_{jm}\,G^{-1}_{mM}\,T_{mp}\,. \tag{10}$$

In the factorial situation, F'_{jm} is known from previous analysis, G_{mM} is given from a knowledge of F'_{jm}, and G^{-1}_{mM} may be computed by Aitkens'* procedure or by Tucker's modification,† if equation (6) does not hold, or from equation (8) if (6) and (7) be true. T_{mp} is known from the previous analysis, and V_{tp} may be computed directly from equation (10).

If, as may frequently be the case, there are more additional tests than there are factors ($s > r$), then it will be computationally simpler to define a new matrix.

$$U_{mp} \equiv G^{-1}_{mM}\,T_{mp} \tag{11}$$

* Thompson, G. H., "Some points of mathematical technique in the factorial analysis of ability," *Jour. Educ. Psychol.*, **27**, 1936, 37-54.

† Tucker, L. R. "A method for finding the inverse of a matrix," *Psychometrika*, **3**, 1938, 189-197.

and to compute V_{jp} from

$$V_{jp} = R'_{tj} F'_{jm} U_{mp} . \qquad (12)$$

If, as may well happen, it is desired to modify the resulting system of primary traits in the light of the primary trait loadings of the additional tests, this may be accomplished by the use of any of the standard procedures.

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PROGRAM OF THE PSYCHOMETRIC SOCIETY

Thursday, September 8, 1938

OHIO STATE UNIVERSITY, COLUMBUS, OHIO

A RATIONAL COMPARISON OF ITEM-SELECTION TECHNIQUES. Dorothy C. Adkins, *University of Chicago.*

A STATISTICAL APPROACH TO TYPOLOGY. Joseph Zubin, *Mental Hospital Survey Committee*, New York City.

RATER'S MINIMAL DISCRIMINATION AS A CRITERION FOR DETERMINING THE OPTIMAL REFINEMENT FOR A RATING SCALE. Horace Champney and Helen Marshall, *Antioch College.*

THE EQUIVALENCE OF THE GENERAL FACTORS FOUND FOR SUCCESSIVE LEVELS ON THE NEW STANFORD REVISION. Quinn McNemar, *Stanford University*

A NEW THEORY OF TRANSFER APPLIED TO EXPERIMENTS IN VISUAL DISCRIMINATION. Harold Gulliksen, *University of Chicago.*

MENTAL FACTORS OF NO IMPORTANCE. Truman L. Kelley, *Harvard University.*

MAXIMUM PREDICTION WITH A MINIMUM NUMBER OF VARIABLES. Robert J. Wherry, *University of North Carolina.*

ACQUIESCENCE AS A FACTOR IN THE MEASUREMENT OF PERSONALITY. Theo. F. Lentz, *Washington University.*

MULTIDIMENSIONAL PSYCHOPHYSICS. M. W. Richardson, *University of Chicago.*

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2. FRUSTRATION AS AN EXPERIMENTAL PROBLEM. Saul Rosenzweig, *Chairman.* (Discussion participants: Roger Barker, Stuart W. Cook, Quin F. Curtis, George Haslerud, J. McV. Hunt, Carlyle F. Jacobsen, O. H. Mowrer, and B. F. Skinner.)

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